Engineering Mechanics

Introduction

Mechanics is the physical science which deals with the effects of forces on objects. No other subject plays a greater role in engineering analysis than mechanics. Although the principles of mechanics are few, they have wide application in engineering. The principles of mechanics are central to research and development in the fields of vibrations, stability and strength of structures and machines, robotics, rocket and

spacecraft design, automatic control, engine performance, fluid flow, electrical machines and apparatus, and molecular, atomic, and subatomic behavior. A thorough understanding of this subject is an essential prerequisite for work in these and many other fields. Mechanics is the oldest of the physical sciences. The early history of

this subject is synonymous with the very beginnings of engineering. The earliest recorded writings in mechanics are those of Archimedes (287–212 B.C.) on the principle of the lever and the principle of buoyancy. Substantial progress came later with the formulation of the laws of vector combination of forces by Stevinus (1548–1620), who also formulated most of the principles of statics. The first investigation of a dynamicsproblem is credited to Galileo (1564–1642) for his experiments with falling stones. The accurate formulation of the laws of motion, as well as the law of gravitation, was made by Newton (1642–1727), who also conceived the idea of the infinitesimal in mathematical analysis. Substantial contributions to the development of mechanics were also made by da Vinci, Varignon, Euler, D'Alembert, Lagrange, Laplace, and others.

Basic Concepts

The following concepts and definitions are basic to the study of mechanics, and they should be understood at the outset.

Space is the geometric region occupied by bodies whose positions are described by linear and angular measurements relative to a coordinate system. For three-dimensional problems, three independent coordinates are needed. For two-dimensional problems, only two coordinates are required.

Time is the measure of the succession of events and is a basic quantity in dynamics. Time is not directly involved in the analysis of statics problems.

Mass is a measure of the inertia of a body, which is its resistance to a change of velocity. Mass can also be thought of as the quantity of matter in a body. The mass of a body affects the gravitational attraction force between it and other bodies. This force appears in many applications in statics.

Force is the action of one body on another. A force tends to move a body in the direction of its action. The action of a force is characterized by its magnitude, by the direction of its action, and by its point of application. Thus force is a vector quantity.

A particle is a body of negligible dimensions. In the mathematical sense, a particle is a body whose dimensions are considered to be near zero so that we may analyze it as a mass concentrated at a point. We often choose a particle as a differential element of a body. We may treat a body as a particle when its dimensions are irrelevant to the description of its position or the action of forces applied to it.

Rigid body. A body is considered rigid when the change in distance between any two of its points is negligible for the purpose at hand. For instance, the calculation of the tension in the cable which supports the boom of a mobile crane under load is essentially unaffected by the small internal deformations in the structural members of the boom. For the purpose, then, of determining the external forces which act on the boom, we may treat it as a rigid body. Statics deals primarily with the calculation of external forces which act on rigid bodies in equilibrium.

Determination of the internal deformations belongs to the study of the mechanics of deformable bodies, which normally follows statics in the curriculum.

Rigid-body Mechanics

• a basic requirement for the study of the mechanics of deformable bodies and the mechanics of fluids (advanced courses).

• essential for the design and analysis of many types of structural members, mechanical components, electrical devices, etc, encountered in engineering.

A rigid body does not deform under load!

Engineering mechanics

- Deals with effect of forces on objects Mechanics principles used in vibration, spacecraft design, fluid flow, electrical, mechanical m/c design etc.

Statics: deals with effect of force on bodies which are not moving

Dynamics: deals with force effect on moving bodies

We consider RIGID BODIES – Non deformable

Scalar quantity: Only magnitude; time, volume, speed, density, mass…

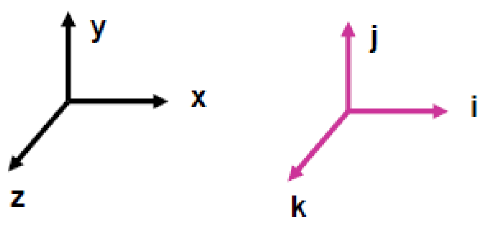
Vector quantity: Both direction and magnitude; Force, displacement, velocity, acceleration, moment…

$V = |v|\, n$, where $|v|$ = magnitude, n = unit vector

$n = V / |v|$

n - dimensionless and in direction of vector 'V'

In our course:

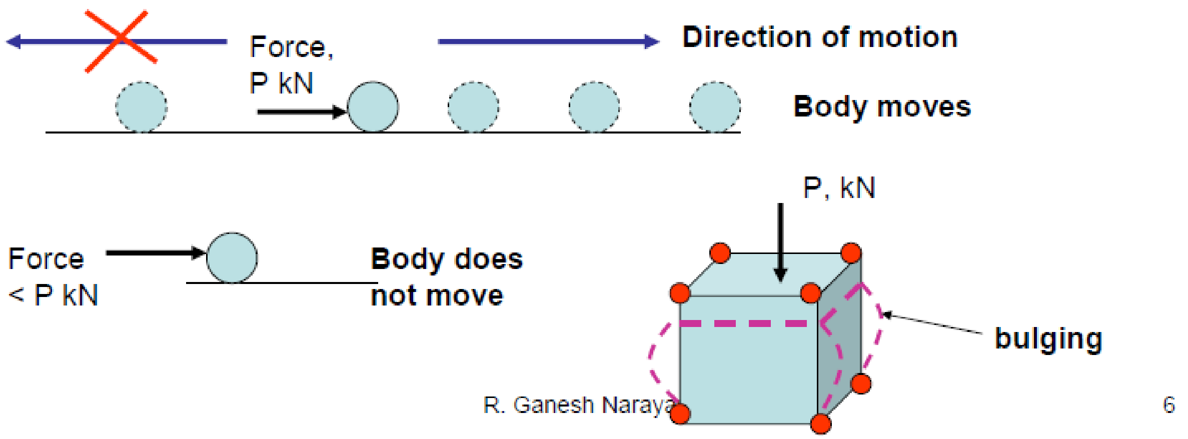


i, j, k – unit vectors

-

Force:

- action of one body on another
- required force can move a body in the direction of action, otherwise no effect
- some times plastic deformation, failure is possible
- Magnitude, direction, point of application; VECTOR



TWO-DIMENSIONAL FORCE SYSTEMS

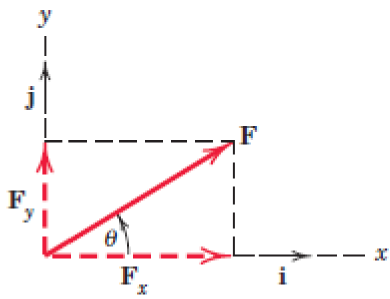
Rectangular Components

The most common two-dimensional resolution of a force vector is into rectangular components. It follows from the parallelogram rule that the vector **F** of Fig. 1 may be written as

$$\mathbf{F} = \mathbf{F}_x + \mathbf{F}_y \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 1$$

where **F***x* and **F***y* are *vector components* of **F** in the *x*- and *y*-directions. Each of the two vector components may be written as a scalar times the

Fig. 1



appropriate unit vector. In terms of the unit vectors i and j of Fig. 1, $F_x = F_{xi}$ and

$F_y = F_{y\,j}$, and thus we may write

$$\mathbf{F} = F_x\mathbf{i} + F_y\mathbf{j}$$

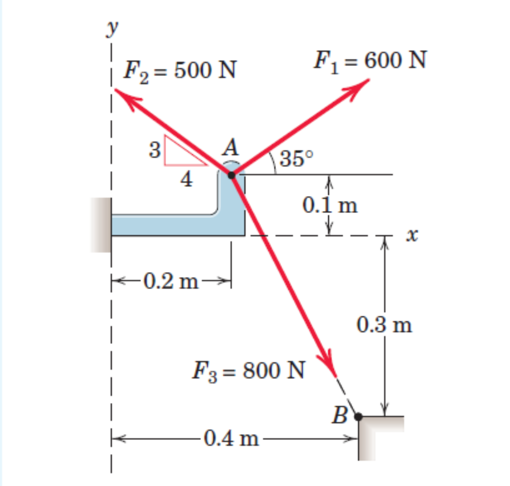
where the scalars F_x and F_y are the x and y scalar components of the vector F. The scalar components can be positive or negative, depending on the quadrant into which F points. For the force vector of Fig. 1, the x and y scalar components are both positive and are related to the magnitude and direction of F by

$$F_x = F\cos\theta \qquad F = \sqrt{F_x^{\,2} + F_y^{\,2}}$$

$$F_y = F\sin\theta \qquad \theta = \tan^{-1}\frac{F_y}{F_x}$$

SAMPLE PROBLEM 2/1

The forces $\mathbf{F}_1$, $\mathbf{F}_2$, and $\mathbf{F}_3$, all of which act on point A of the bracket, are specified in three different ways. Determine the x and y scalar components of each of the three forces.



Solution. The scalar components of $\mathbf{F}_1$, from Fig. a, are

$$F_{1_x} = 600 \cos 35° = 491 \text{ N} \qquad \textit{Ans.}$$

$$F_{1_y} = 600 \sin 35° = 344 \text{ N} \qquad \textit{Ans.}$$

The scalar components of $\mathbf{F}_2$, from Fig. b, are

$$F_{2_x} = -500(\tfrac{4}{5}) = -400 \text{ N} \qquad \textit{Ans.}$$

$$F_{2_y} = 500(\tfrac{3}{5}) = 300 \text{ N} \qquad \textit{Ans.}$$

Note that the angle which orients $\mathbf{F}_2$ to the x-axis is never calculated. The cosine and sine of the angle are available by inspection of the 3-4-5 triangle. Also note that the x scalar component of $\mathbf{F}_2$ is negative by inspection.

The scalar components of $\mathbf{F}_3$ can be obtained by first computing the angle α of Fig. c.

$$\alpha = \tan^{-1}\left[\frac{0.2}{0.4}\right] = 26.6°$$

1 Then,

$$F_{3_x} = F_3 \sin \alpha = 800 \sin 26.6° = 358 \text{ N} \qquad \textit{Ans.}$$

$$F_{3_y} = -F_3 \cos \alpha = -800 \cos 26.6° = -716 \text{ N} \qquad \textit{Ans.}$$

Alternatively, the scalar components of $\mathbf{F}_3$ can be obtained by writing $\mathbf{F}_3$ as a magnitude times a unit vector $\mathbf{n}_{AB}$ in the direction of the line segment AB. Thus,

2

$$\mathbf{F}_3 = F_3\mathbf{n}_{AB} = F_3 \frac{\overrightarrow{AB}}{\overline{AB}} = 800\left[\frac{0.2\mathbf{i} - 0.4\mathbf{j}}{\sqrt{(0.2)^2 + (-0.4)^2}}\right]$$

$$= 800\,[0.447\mathbf{i} - 0.894\mathbf{j}]$$

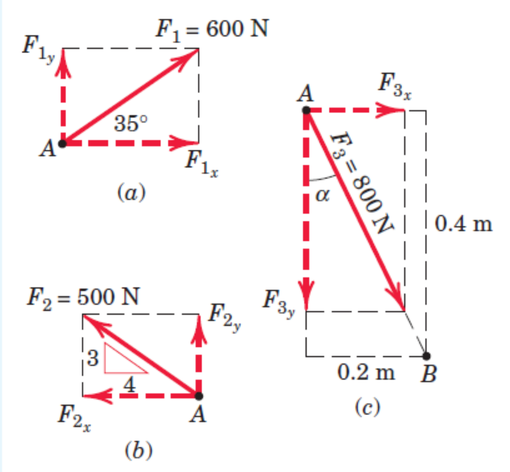
$$= 358\mathbf{i} - 716\mathbf{j} \text{ N}$$

The required scalar components are then

$$F_{3_x} = 358 \text{ N} \qquad \textit{Ans.}$$

$$F_{3_y} = -716 \text{ N} \qquad \textit{Ans.}$$

which agree with our previous results.



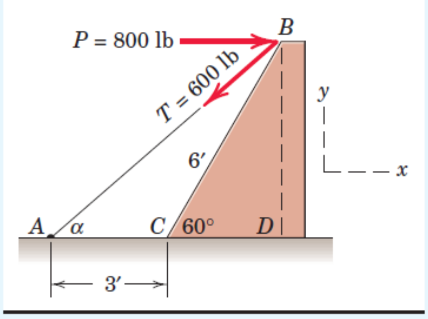
Helpful Hints

1 You should carefully examine the geometry of each component determination problem and not rely on the blind use of such formulas as $F_x = F \cos \theta$ and $F_y = F \sin \theta$.

2 A unit vector can be formed by dividing *any* vector, such as the geometric position vector $\overrightarrow{AB}$, by its length or magnitude. Here we use the overarrow to denote the vector which runs from A to B and the overbar to determine the distance between A and B.

SAMPLE PROBLEM 2/2

Combine the two forces **P** and **T**, which act on the fixed structure at B, into a single equivalent force **R**.

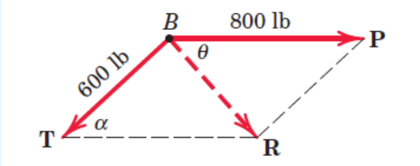


Graphical solution. The parallelogram for the vector addition of forces **T** and **P** is constructed as shown in Fig. *a*. The scale used here is 1 in. = 800 lb; a scale of 1 in. = 200 lb would be more suitable for regular-size paper and would give greater accuracy. Note that the angle α must be determined prior to construction of the parallelogram. From the given figure

$$\tan \alpha = \frac{\overline{BD}}{\overline{AD}} = \frac{6 \sin 60^\circ}{3 + 6 \cos 60^\circ} = 0.866 \qquad \alpha = 40.9^\circ$$

Measurement of the length R and direction θ of the resultant force **R** yields the approximate results

$$R = 525 \text{ lb} \qquad \theta = 49^\circ \qquad \textit{Ans.}$$



(*a*)

Geometric solution. The triangle for the vector addition of **T** and **P** is shown in Fig. *b*. The angle α is calculated as above. The law of cosines gives

$$R^2 = (600)^2 + (800)^2 - 2(600)(800) \cos 40.9^\circ = 274{,}300$$

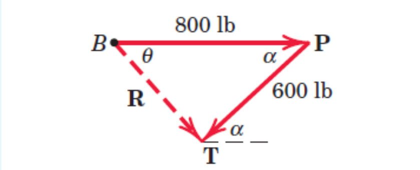
$$R = 524 \text{ lb} \qquad \textit{Ans.}$$

From the law of sines, we may determine the angle θ which orients **R**. Thus,

$$\frac{600}{\sin \theta} = \frac{524}{\sin 40.9^\circ} \qquad \sin \theta = 0.750 \qquad \theta = 48.6^\circ \qquad \textit{Ans.}$$

Helpful Hints

1. Note the repositioning of **P** to permit parallelogram addition at B.



(*b*)

2. Note the repositioning of **F** so as to preserve the correct line of action of the resultant **R**.

Algebraic solution. By using the x-y coordinate system on the given figure, we may write

$$R_x = \Sigma F_x = 800 - 600 \cos 40.9^\circ = 346 \text{ lb}$$

$$R_y = \Sigma F_y = -600 \sin 40.9^\circ = -393 \text{ lb}$$

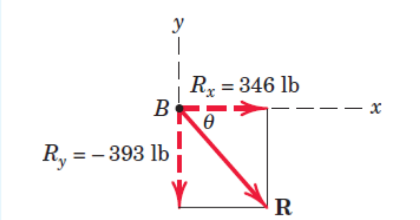
The magnitude and direction of the resultant force **R** as shown in Fig. *c* are then

$$R = \sqrt{R_x{}^2 + R_y{}^2} = \sqrt{(346)^2 + (-393)^2} = 524 \text{ lb} \qquad \textit{Ans.}$$

$$\theta = \tan^{-1} \frac{|R_y|}{|R_x|} = \tan^{-1} \frac{393}{346} = 48.6^\circ \qquad \textit{Ans.}$$

The resultant **R** may also be written in vector notation as

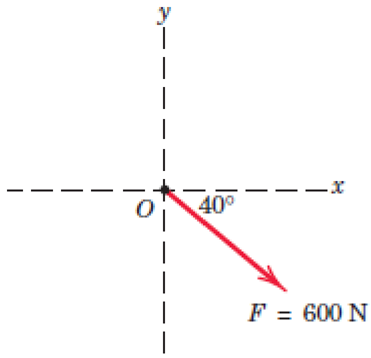
$$\mathbf{R} = R_x\mathbf{i} + R_y\mathbf{j} = 346\mathbf{i} - 393\mathbf{j} \text{ lb} \qquad \textit{Ans.}$$



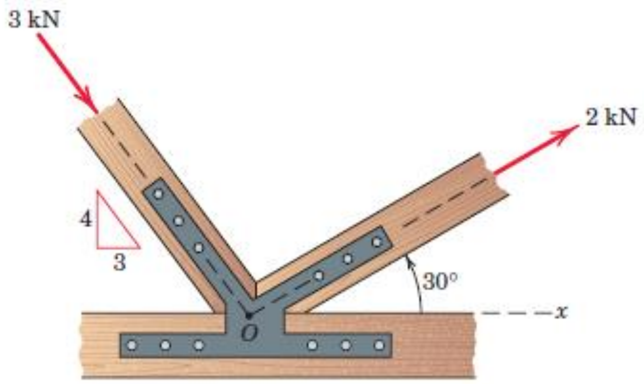
(*c*)

PROBLEMS:-

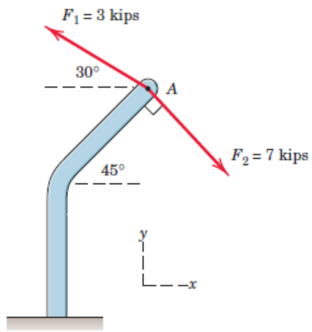
Q1 / 2/1 The force F has a magnitude of 600 N. Express F as a vector in terms of the unit vectors i and j. Identify the x and y scalar components of F.



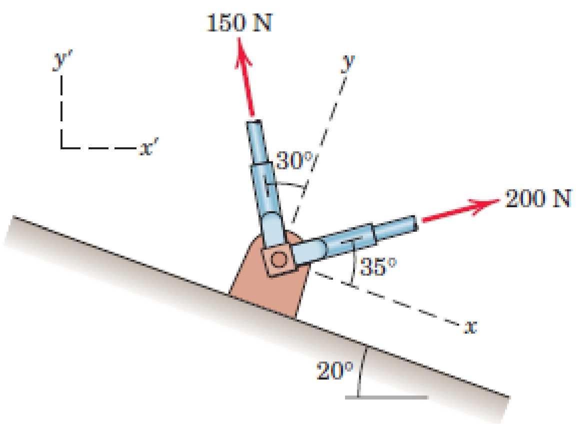
Q2/ The two structural members, one of which is in tension and the other in compression, exert the indicated forces on joint O. Determine the magnitude of the resultant R of the two forces and the angle Which R makes with the positive x-axis.



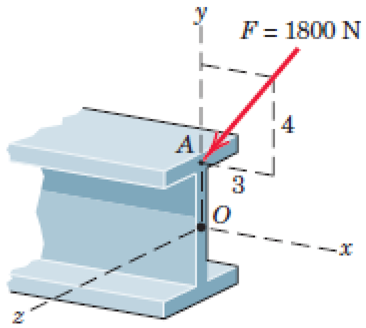
Q3/ The two forces shown act at point A of the bent bar. Determine the resultant R of the two forces.



Q4/ Determine the resultant R of the two forces applied to the bracket. Write R in terms of unit vectors along the x- and y-axes shown.

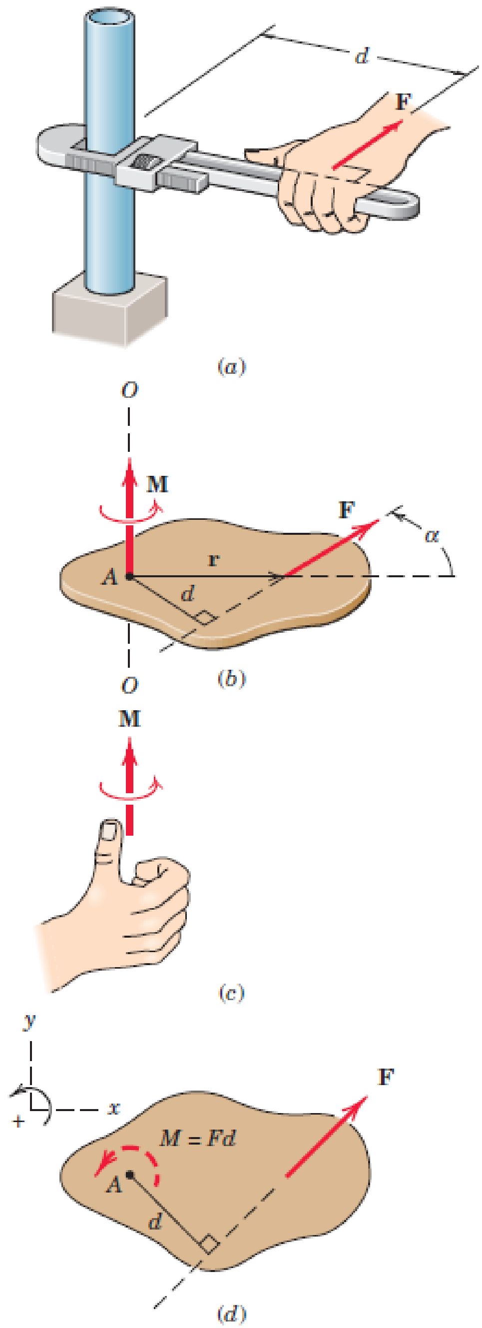


Q5/ The 1800-N force F is applied to the end of the I-beam. Express F as a vector using the unit vectors i and j.



Moment

In addition to the tendency to move a body in the direction of its application, a force can also tend to rotate a body about an axis. The axis may be any line which neither intersects nor is parallel to the line of action of the force. This rotational tendency is known as the *moment* **M** of the force. Moment is also referred to as *torque*. As a familiar example of the concept of moment, consider the pipe wrench of Fig. 2/8*a*. One effect of the force applied perpendicular to the handle of the wrench is the tendency to rotate the pipe about its vertical axis. The magnitude of this tendency depends on both the magnitude *F* of the force and the effective length *d* of the wrench handle. Common experience shows that a pull which is not perpendicular to the wrench handle is less effective than the right-angle pull shown.



Moment about a Point

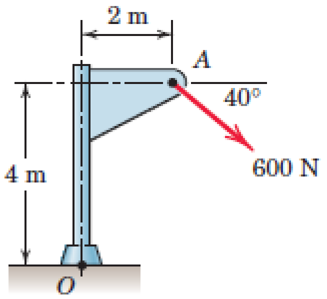
Figure 2b shows a two-dimensional body acted on by a force F in its plane. The magnitude of the moment or tendency of the force to rotate the body about the axis O-O perpendicular to the plane of the body is proportional both to the magnitude of the force and to the moment arm d, which is the perpendicular distance from the axis to the line of action of the force. Therefore, the magnitude of the moment is defined as

$$M = Fd$$

The moment is a vector M perpendicular to the plane of the body. The sense of M depends on the direction in which F tends to rotate the body. The right-hand rule, Fig. 2c, is used to identify this sense. We represent the moment of F about O-O as a vector pointing in the direction of the thumb, with the fingers curled in the direction of the rotational tendency. The moment M obeys all the rules of vector combination and may be considered a sliding vector with a line of action coinciding with the moment axis. The basic units of moment in SI units are Newton-meters (M.m) and in the U.S. customary system are pound-feet (lb-ft). When dealing with forces which all act in a given plane, we customarily speak of the moment about a point. By this we mean the moment with respect to an axis normal to the plane and passing through the point. Thus, the moment of force F about point A in Fig. 2d has the magnitude M = Fd and is counterclockwise. Moment directions may be accounted for by using a stated sign convention, such as a plus sign (+) for counterclockwise moments and a minus sign (-) for clockwise moments, or vice versa. Sign consistency within a given problem is essential. For the sign convention of Fig. 2d, the moment of F about point A (or about the z-axis passing through point A) is positive. The curved arrow of the figure is a convenient way to represent moments in two-dimensional analysis.

SAMPLE PROBLEM

Example1 /Calculate the magnitude of the moment about the base point O of the 600-N Force in five different ways.

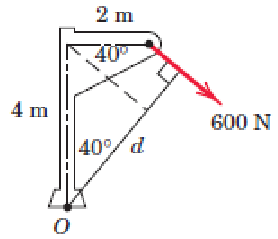


Solution. *(I)* The moment arm to the 600-N force is

$$d = 4 \cos 40° + 2 \sin 40° = 4.35 \text{ m}$$

1 By $M = Fd$ the moment is clockwise and has the magnitude

$$M_O = 600(4.35) = 2610 \text{ N·m} \qquad \textit{Ans.}$$

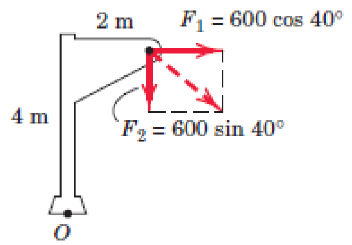


(II) Replace the force by its rectangular components at A,

$$F_1 = 600 \cos 40° = 460 \text{ N}, \qquad F_2 = 600 \sin 40° = 386 \text{ N}$$

By Varignon's theorem, the moment becomes

2 $$M_O = 460(4) + 386(2) = 2610 \text{ N·m} \qquad \textit{Ans.}$$

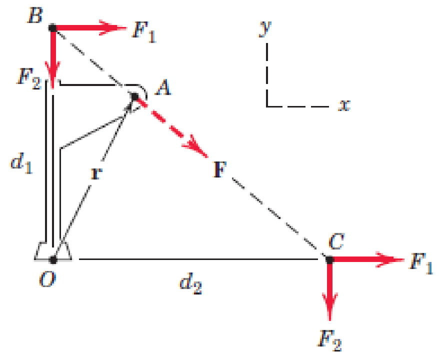


(III) By the principle of transmissibility, move the 600-N force along its line of action to point B, which eliminates the moment of the component F_2. The moment arm of F_1 becomes

$$d_1 = 4 + 2 \tan 40° = 5.68 \text{ m}$$

and the moment is

$$M_O = 460(5.68) = 2610 \text{ N·m} \qquad \textit{Ans.}$$



3 *(IV)* Moving the force to point C eliminates the moment of the component F_1. The moment arm of F_2 becomes

$$d_2 = 2 + 4 \cot 40° = 6.77 \text{ m}$$

and the moment is

$$M_O = 386(6.77) = 2610 \text{ N·m} \qquad \textit{Ans.}$$

(V) By the vector expression for a moment, and by using the coordinate system indicated on the figure together with the procedures for evaluating cross products, we have

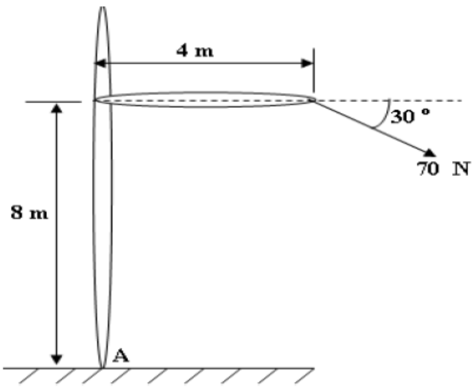
4 $$\mathbf{M}_O = \mathbf{r} \times \mathbf{F} = (2\mathbf{i} + 4\mathbf{j}) \times 600(\mathbf{i} \cos 40° - \mathbf{j} \sin 40°)$$
$$= -2610\mathbf{k} \text{ N·m}$$

The minus sign indicates that the vector is in the negative z-direction. The magnitude of the vector expression is

$$M_O = 2610 \text{ N·m} \qquad \textit{Ans.}$$

Ex (4) :

Determine the moment of the force (**70 N**) shown in fig. about the Point (**A**) .



Solution

Fx = F . cos θ = 70 cos 30
= 70 * 0.866 = 60.62 N

Fy = F . sin θ = 70 sin 30
= 70 * 0.5 = 35 N

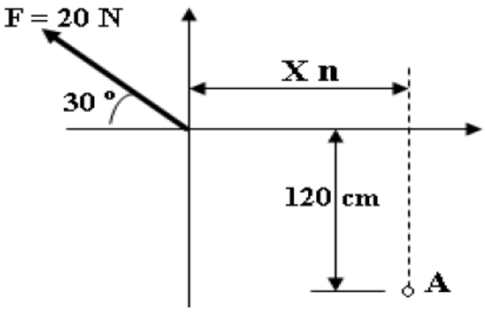
M1 = Fx * d = 60.62 * 8 = 484.97 N . m

M2 = Fy * d = 35 * 4 = 140 N . m

M (A) = M1 + M2 = 484.97 N + 140 = 624.97 N . m

Ex (5) :

Find the distance (**Xn**) , if the moment of the force (**F**) about the point (**A**) is equal to **zero** .



Solution

Fx = F . cos θ = 20 cos 30
= 20 * 0.866 = 17.32 N

Fy = F . sin θ = 20 sin 30
= 20 * 0.5 = 10 N

M1 = Fx * d = 17.32 * 120 = **-** 2078.46 N . cm

M2 = Fy * d = 10 * Xn = 10 Xn N .cm

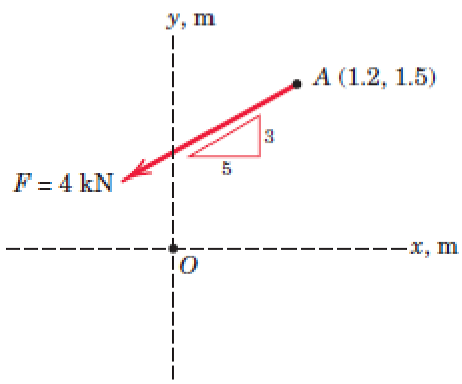
M (A) = - M1+ M2

0 = **-** 2078.46 + 10 Xn

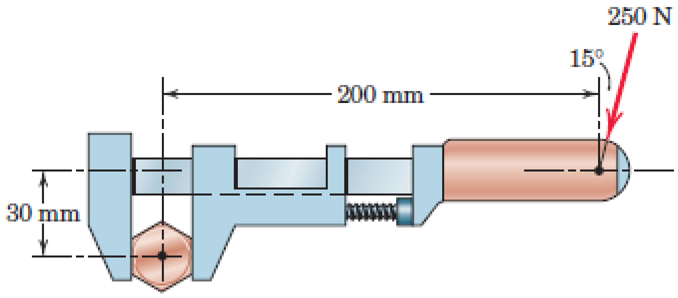
Xn = 2078.46 / 10 = 207.84 cm

PROBLEMS

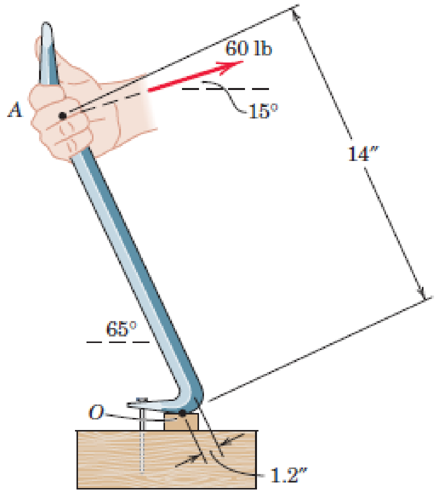
1- The 4-kN force F is applied at point A. Compute the moment of F about point O, expressing it both as a scalar and as a vector quantity. Determine the coordinates of the points on the x- and y-axes about which the moment of F is zero.



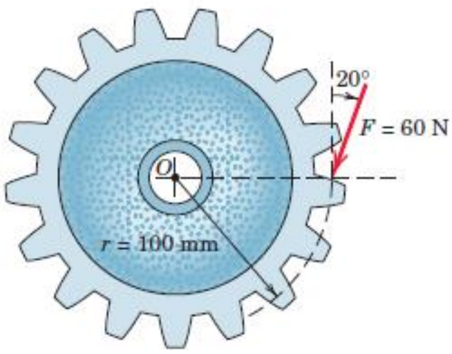
2- Calculate the moment of the 250-N force on the handle of the monkey wrench about the center of the bolt.



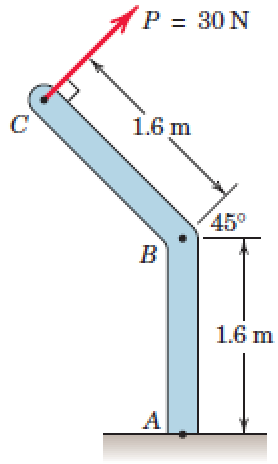
3- A prybar is used to remove a nail as shown. Determine the moment of the 60-lb force about the point O of contact between the prybar and the small support block.



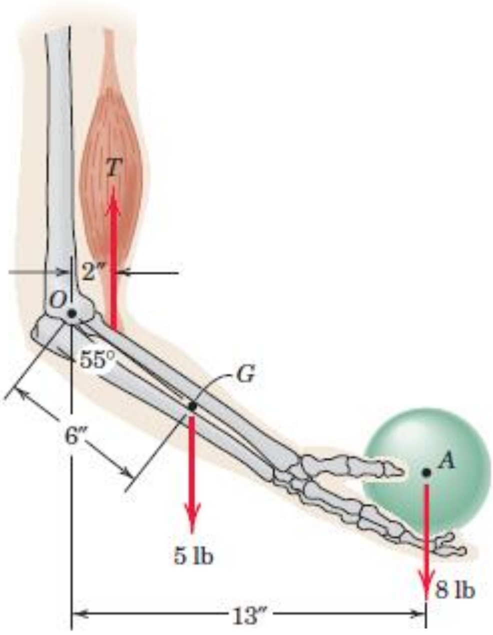
4- A force F of magnitude 60 N is applied to the gear. Determine the moment of F about point O.



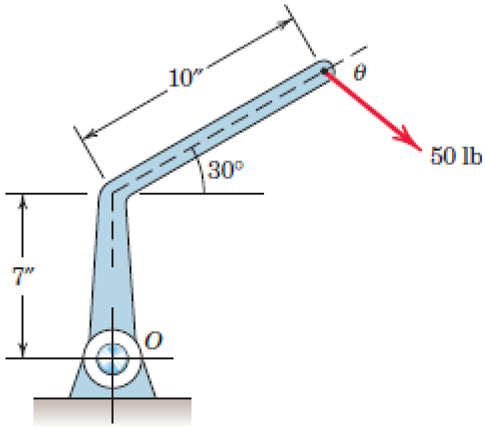
5- The 30-N force P is applied perpendicular to the portion BC of the bent bar. Determine the moment of P about point B and about point A.



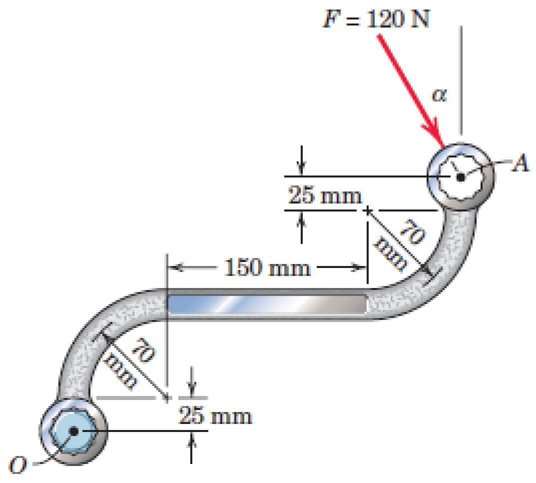
6- Elements of the lower arm are shown in the figure. The weight of the forearm is 5 lb with mass center at G. Determine the combined moment about the elbow pivot O of the weights of the forearm and the sphere. What must the biceps tension force be so that the overall moment about O is zero?



7- Determine the angle θ which will maximize the moment Mo of the 50-lb force about the shaft axis at O. Also compute Mo.



8- The 120-N force is applied as shown to one end of the curved wrench. If α= 30 º calculate the moment of F about the center O of the bolt. Determine the value of which would maximize the moment about O; state the value of this maximum moment.

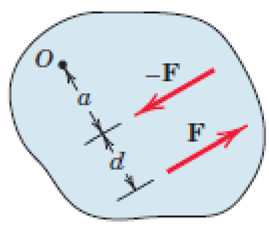


Couple

The moment produced by two equal, opposite, and no collinear forces is called a couple. Couples have certain unique properties and have important applications in mechanics. Consider the action of two equal and opposite forces F and -F a distance d apart, as shown in Fig a.. These two forces cannot be combined into a single force because their sum in every direction is zero. Their only effect is to produce a tendency of rotation. The combined moment of the two forces about an axis normal to their plane and passing through any point such as O in their plane is the couple M. This couple has a magnitude

$$M = F(a + d) - Fa$$

$$M = Fd$$



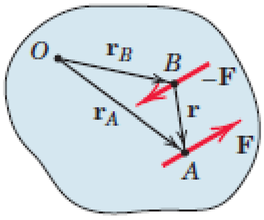
(a)

Its direction is counterclockwise when viewed from above for the case illustrated.

Note especially that the magnitude of the couple is independent of the distance a which locates the forces with respect to the moment center O. It follows that the moment of a couple has the same value for all moment centers.

Vector Algebra Method:-

We may also express the moment of a couple by using vector algebra. With the cross-product notation of Eq. 2/6, the combined moment about point O of the forces forming the couple of Fig. b is



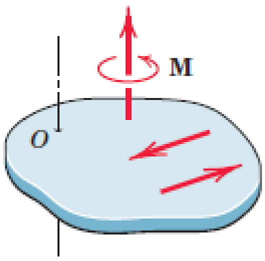
(b)

$$\mathbf{M} = \mathbf{r}_A \times \mathbf{F} + \mathbf{r}_B \times (-\mathbf{F}) = (\mathbf{r}_A - \mathbf{r}_B) \times \mathbf{F}$$

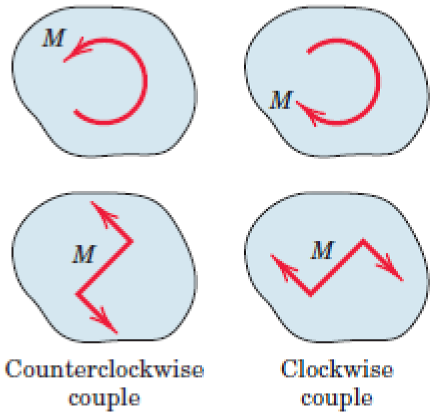
where r_A and r_B are position vectors which run from point O to arbitrary points A and B on the lines of action of F and -F, respectively. Because $r_A - r_B = r$, we can express M as

$$M = r \times F$$

Here again, the moment expression contains no reference to the moment center O and, therefore, is the same for all moment centers. Thus, we may represent M by a free vector, as shown in Fig. c, where the direction of M is normal to the plane of the couple and the sense of M is established by the right-hand rule. Because the couple vector M is always perpendicular to the plane of the forces which constitute the couple, in two-dimensional analysis we can represent the sense of a couple vector as clockwise or counterclockwise by one of the conventions shown in Fig d. Later, when we deal with couple vectors in three-dimensional problems, we will make full use of vector notation to represent them, and the mathematics will automatically account for their sense.



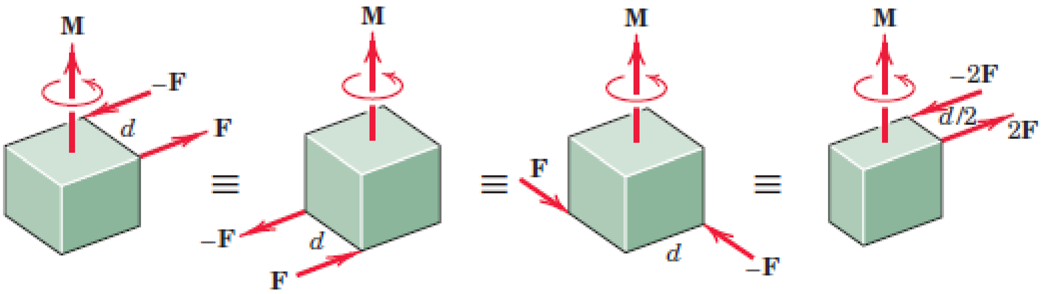
(c)



(d)

Equivalent Couples:-

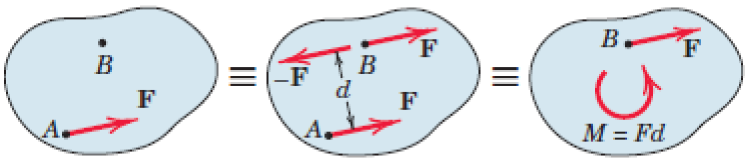
Changing the values of F and d does not change a given couple as long as the product F d remains the same. Likewise, a couple is not affected if the forces act in a different but parallel plane. Figure shows four different configurations of the same couple M. In each of the four cases, the couples are equivalent and are described by the same free vector which represents the identical tendencies to rotate the bodies.



Force–Couple Systems:-

The effect of a force acting on a body is the tendency to push or pull the body in the direction of the force, and to rotate the body about any fixed axis which does not intersect the line of the force. We can represent this dual effect more easily by replacing the given force by an equal parallel force and a couple to compensate for the change in the moment of the force. The replacement of a force by a force and a couple is illustrated in Fig., where the given force F acting at point A is replaced by an equal force F at some point B and the counterclockwise couple M = F d. The transfer is seen in the middle figure, where the equal and opposite forces F and -F are added at point B without introducing any net external effects on the body. We now see that the original force at A and the equal and opposite one at B constitute the couple M = F d, which is

Counterclockwise for the sample chosen, as shown in the right-hand part of the figure. Thus, we have replaced the original force at A by the same force acting at a different point B and a couple, without altering the external effects of the original force on the body. The combination of the force and couple in the right-hand part of Fig. is referred to as a force–couple system. By reversing this process, we can combine a given couple and a force which lies in the plane of the couple (normal to the couple vector) to produce a single, equivalent force. Replacement of a force by an equivalent force–couple system, and the reverse procedure, have many applications in mechanics and should be mastered.



SAMPLE PROBLEM:-

Q1/ The rigid structural member is subjected to a couple consisting of the two 100-N forces. Replace this couple by an equivalent couple consisting of the two forces P and -P, each of which has a magnitude of 400 N. Determine the proper angle θ.

Solution. The original couple is counterclockwise when the plane of the forces is viewed from above, and its magnitude is

$[M = Fd]$ $\quad M = 100(0.1) = 10\ \text{N}\cdot\text{m}$

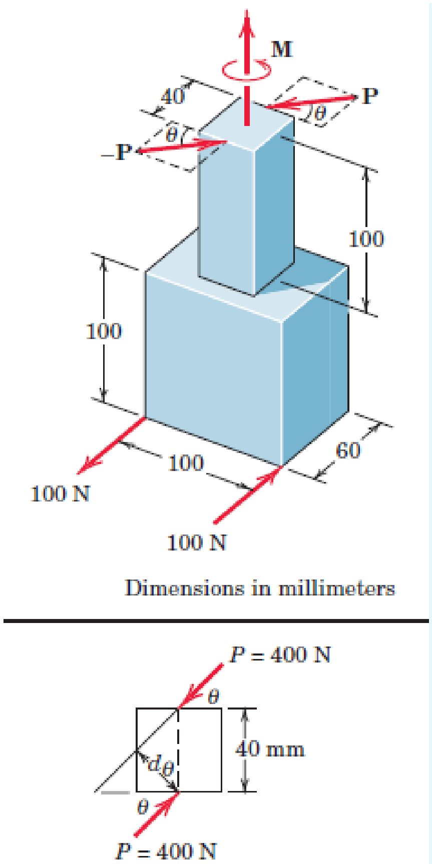
The forces **P** and −**P** produce a counterclockwise couple

$$M = 400(0.040)\cos\theta$$

Equating the two expressions gives

$$10 = (400)(0.040)\cos\theta$$

$$\theta = \cos^{-1}\frac{10}{16} = 51.3° \qquad \textit{Ans.}$$



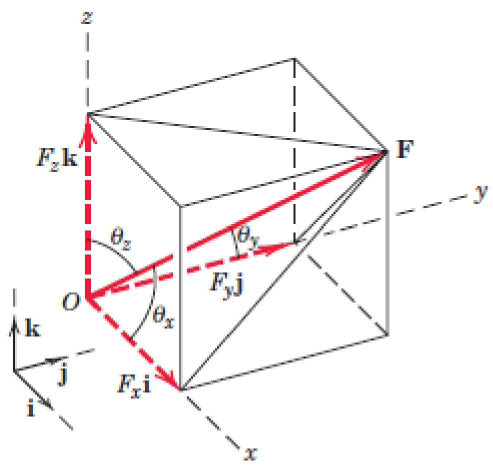
THREE-DIMENSIONAL FORCE SYSTEMS

Many problems in mechanics require analysis in three dimensions, and for such problems it is often necessary to resolve a force into its three mutually perpendicular components. The force F acting at point O in Fig. has the rectangular components F_x, F_y, F_z, where

$$F_x = F\cos\theta_x \qquad F = \sqrt{F_x^2 + F_y^2 + F_z^2}$$

$$F_y = F\cos\theta_y \qquad \mathbf{F} = F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k}$$

$$F_z = F\cos\theta_z \qquad \mathbf{F} = F(\mathbf{i}\cos\theta_x + \mathbf{j}\cos\theta_y + \mathbf{k}\cos\theta_z)$$



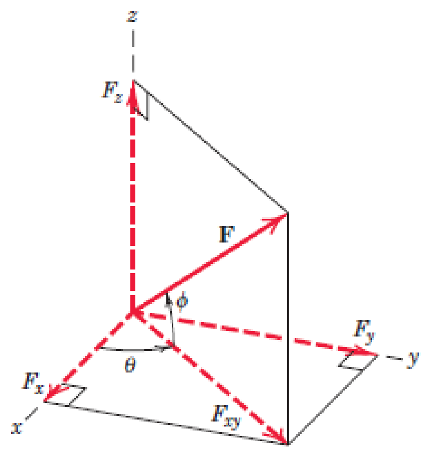
Specification by two angles which orient the line of action of the force. Consider the geometry of Fig. We assume that the angles θ and ϕ are known. First resolve F into horizontal and vertical components.

$$F_{xy} = F\cos\phi$$

$$F_z = F\sin\phi$$

$$F_x = F_{xy}\cos\theta = F\cos\phi\cos\theta$$

$$F_y = F_{xy}\sin\theta = F\cos\phi\sin\theta$$

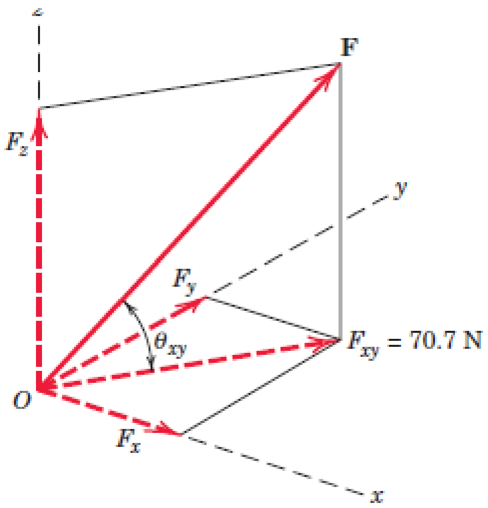


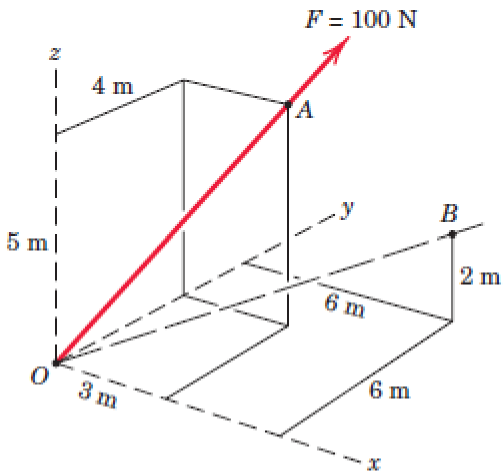
Then resolve the horizontal component F_{xy} into x- and y-components

The quantities F_x, F_y, and F_z are the desired scalar components of F. The choice of orientation of the coordinate system is arbitrary, with convenience being the primary consideration. However, we must use a right-handed set of axes in our three-dimensional work to be consistent with the right-hand-rule definition of the cross product. When we rotate from the x- to the y-axis through the 90 angle, the positive direction for the z-axis in a right-handed system is that of the advancement of a right-handed screw rotated in the same sense. This is equivalent to the right-hand rule.

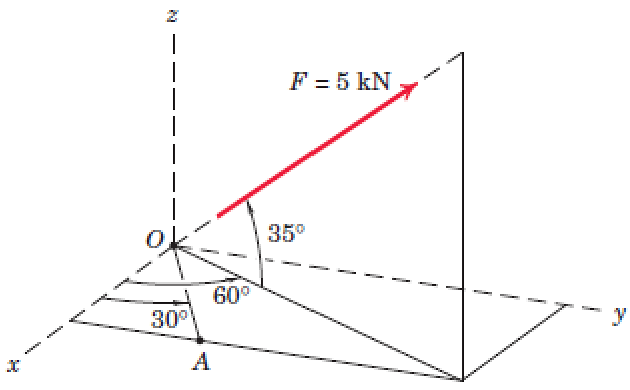
PROBLEM

Q1/A force F with a magnitude of 100 N is applied at the origin O of the axes x-y-z as shown. The line of action of F passes through a point A whose coordinates are 3 m, 4 m, and 5 m. Determine (a) the x, y, and z scalar components of F?

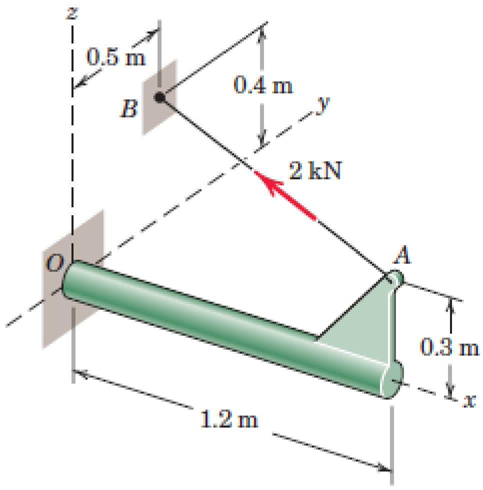




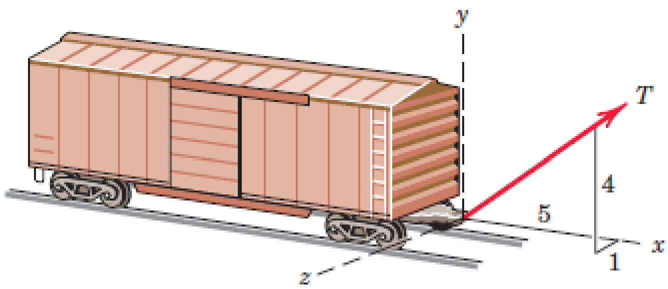
Q2/ Determine (a) the x, y, and z scalar components of F



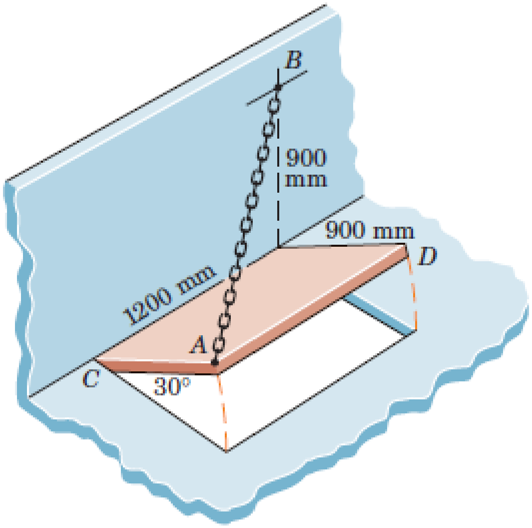
Q3/ The cable exerts a tension of 2 kN on the fixed bracket at A, Determine (a) the x, y, and z scalar components of it



Q4/ An overhead crane is used to reposition the boxcar within a railroad car-repair shop. If the boxcar begins to move along the rails when the x-component of the cable tension reaches 600 lb, calculate the necessary tension T in the cable. Determine the angle θ_{xy} between the cable and the vertical x-y plane.



Q5/ The access door is held in the 30° open position by the chain AB. If the tension in the chain is 100 N, determine the projection of the tension force onto the diagonal axis CD of the door.



Equilibrium

Introduction:-

Statics deals primarily with the description of the force conditions necessary and sufficient to maintain the equilibrium of engineering structures. This chapter on equilibrium, therefore, constitutes the most important part of statics, and the procedures developed here form the basis for solving problems in both statics and dynamics. We will make continual use of the forces, moments, couples, and resultants as we apply the principles of equilibrium. When a body is in equilibrium, the resultant of all forces acting on it is zero. Thus, the resultant force R and the resultant couple M are both zero, and we have the equilibrium equations

$$\mathbf{R} = \Sigma\mathbf{F} = 0 \qquad \mathbf{M} = \Sigma\mathbf{M} = 0$$

These requirements are both necessary and sufficient conditions for equilibrium. All physical bodies are three-dimensional, but we can treat many of them as two-dimensional when the forces to which they are subjected act in a single plane or can be projected onto a single plane. When this simplification is not possible, the problem must be treated as three dimensional. We will follow the arrangement used in Chapter 2, and discuss in Section A the equilibrium of bodies subjected to two-dimensional

EQUILIBRIUM IN TWO DIMENSIONS

System Isolation and the Free-Body Diagram

we must define unambiguously the particular body or mechanical system to be analyzed and represent clearly and completely all forces acting on the body. Omission of a force which acts on the body in question, or inclusion of a force which does not act on the body, will give erroneous results. A mechanical system is defined as a body or group of bodies which can be conceptually isolated from all other bodies. A system may be a single body or a combination of connected bodies. The bodies may be rigid or non rigid. The system may also be an identifiable fluid

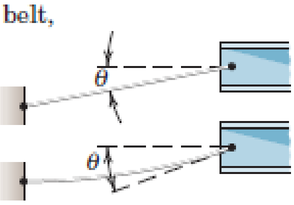
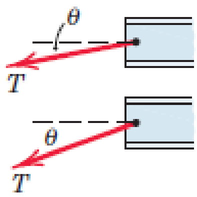
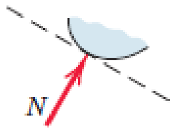
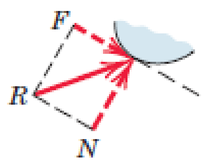
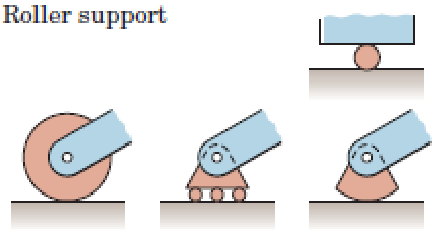
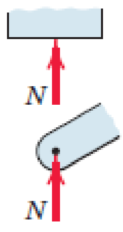
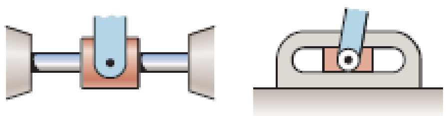
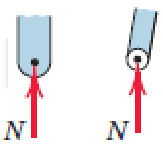
mass, either liquid or gas, or a combination of fluids and solids. In statics we study primarily forces which act on rigid bodies at rest, although we also study forces acting on fluids in equilibrium. Once we decide which body or combination of bodies to analyze, we then treat this body or combination as a single body isolated from all surrounding bodies. This isolation is accomplished by means of the free-body diagram, which is a diagrammatic representation of the isolated system treated as a single body. The diagram shows all forces applied to the system by mechanical contact with other bodies, which are imagined to be removed. If appreciable body forces are present, such as gravitational or magnetic attraction, then these forces must also be shown on the free-body diagram of the isolated system. Only after such a diagram has been carefully drawn should the equilibrium equations be written. Because of its critical importance, we emphasize
here that

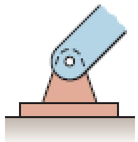
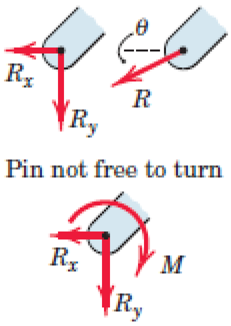
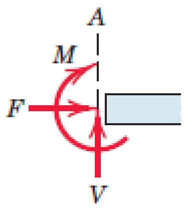
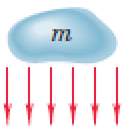
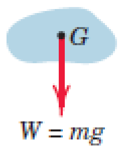
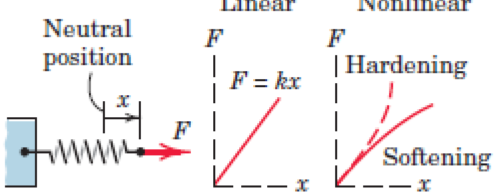
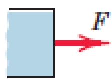
The free-body diagram is the most important single step in
The solution of problems in mechanics.

Before attempting to draw a free-body diagram, we must recall the basic characteristics of force. These characteristics were described in Art. 2/2, with primary attention focused on the vector properties of force. Forces can be applied either by direct physical contact or by remote action. Forces can be either internal or external to the system under consideration. Application of force is accompanied by reactive force, and both applied and reactive forces may be either concentrated or distributed. The principle of transmissibility permits the treatment of force as a sliding vector as far as its external effects on a rigid body are concerned. We will now use these force characteristics to develop conceptual models of isolated mechanical systems. These models enable us to 110 Chapter 3 Equilibrium force systems and in Section B the equilibrium of bodies subjected to three-dimensional force systems

Modeling the Action of Forces

Figure below shows the common types of force application on mechanical systems for analysis in two dimensions. Each example shows the force exerted on the body to be isolated, by the body to be removed. Newton's third law, which notes the existence of an equal and opposite reaction to every action, must be carefully observed. The force exerted on the body in question by a contacting or supporting member is always in the sense to oppose the movement of the isolated body which would occur if the ontacting or supporting body were removed.

MODELING THE ACTION OF FORCES IN TWO-DIMENSIONAL ANALYSIS	
Type of Contact and Force Origin	**Action on Body to Be Isolated**
1. Flexible cable, belt, chain, or rope Weight of cable negligible Weight of cable not negligible 	 Force exerted by a flexible cable is always a tension away from the body in the direction of the cable.
2. Smooth surfaces 	 Contact force is compressive and is normal to the surface.
3. Rough surfaces 	 Rough surfaces are capable of supporting a tangential component F (frictional force) as well as a normal component N of the resultant contact force R.
4. Roller support 	 Roller, rocker, or ball support transmits a compressive force normal to the supporting surface.
5. Freely sliding guide 	 Collar or slider free to move along smooth guides; can support force normal to guide only.

MODELING THE ACTION OF FORCES IN TWO-DIMENSIONAL ANALYSIS (*cont.*)	
Type of Contact and Force Origin	Action on Body to Be Isolated
6. Pin connection 	Pin free to turn; Pin not free to turn  A freely hinged pin connection is capable of supporting a force in any direction in the plane normal to the pin axis. We may either show two components R_x and R_y or a magnitude R and direction θ. A pin not free to turn also supports a couple M.
7. Built-in or fixed support 	 A built-in or fixed support is capable of supporting an axial force F, a transverse force V (shear force), and a couple M (bending moment) to prevent rotation.
8. Gravitational attraction 	 The resultant of gravitational attraction on all elements of a body of mass m is the weight $W = mg$ and acts toward the center of the earth through the center mass G.
9. Spring action 	 Spring force is tensile if spring is stretched and compressive if compressed. For a linearly elastic spring the stiffness k is the force required to deform the spring a unit distance.

Construction of Free-Body Diagrams

The full procedure for drawing a free-body diagram which isolates a body or system consists of the following steps.

Step 1. Decide which system to isolate. The system chosen should usually involve one or more of the desired unknown quantities.

Step 2. Next isolate the chosen system by drawing a diagram which represents its *complete external boundary*. This boundary defines the isolation of the system from *all* other attracting or contacting bodies, which are considered removed. This step is often the most crucial of all. Make certain that you have *completely isolated* the system before proceeding with the next step.

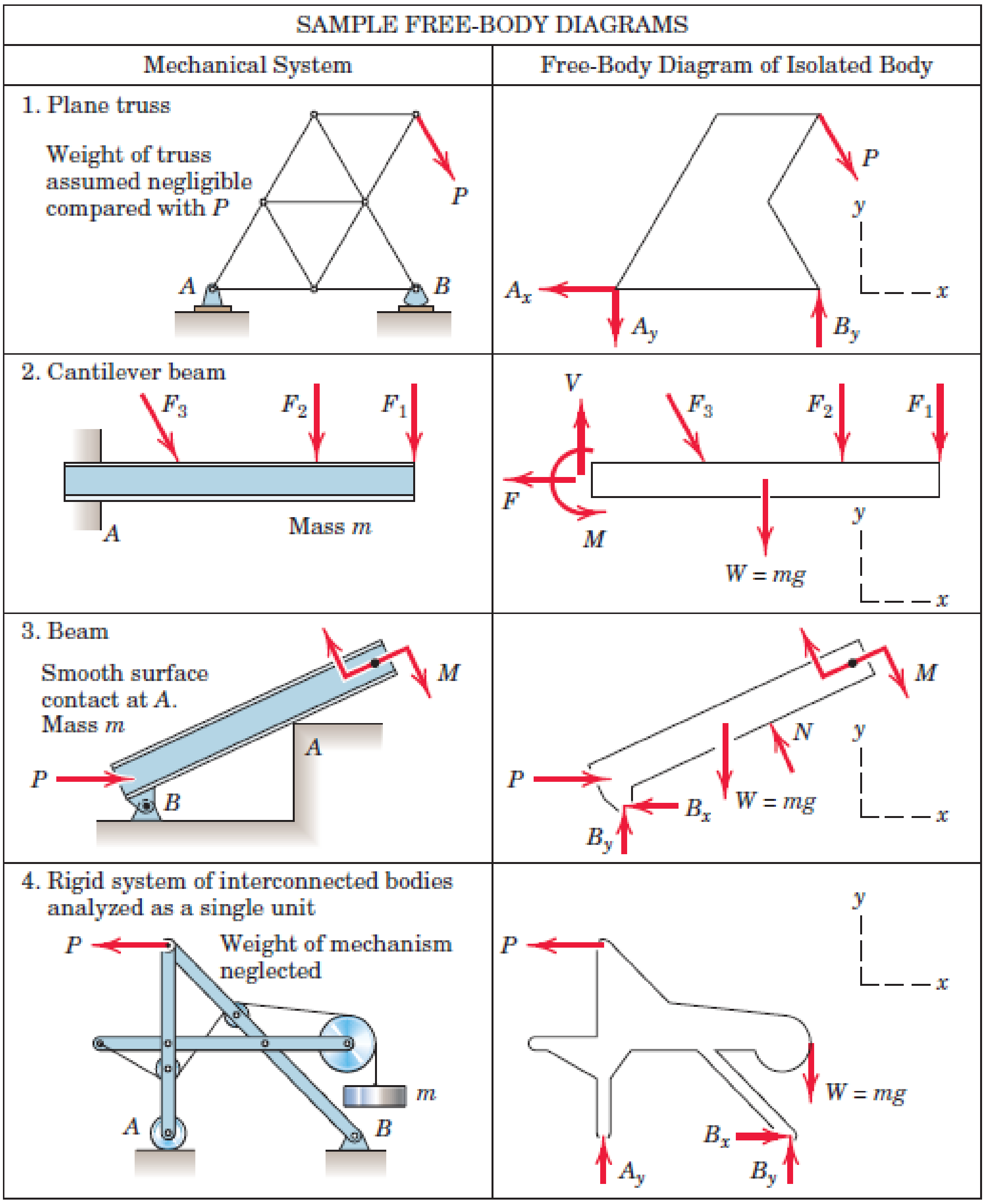
Step 3. Identify all forces which act *on* the isolated system as applied *by* the removed contacting and attracting bodies, and represent them in their proper positions on the diagram of the isolated system. Make a systematic traverse of the entire boundary to identify all contact forces. Include body forces such as weights, where appreciable. Represent all known forces by vector arrows, each with its proper magnitude, direction, and sense indicated. Each unknown force should be represented by a vector arrow with the unknown magnitude or direction indicated by symbol. If the sense of the vector is also unknown, you must arbitrarily assign a sense. The subsequent calculations with the equilibrium equations will yield a positive quantity if the correct sense was assumed and a negative quantity if the incorrect sense was assumed. It is

necessary to be *consistent* with the assigned characteristics of unknown forces throughout all of the calculations. If you are consistent, the solution of the equilibrium equations will reveal the correct senses.

Step 4. Show the choice of coordinate axes directly on the diagram. Pertinent dimensions may also be represented for convenience. Note, however, that the free-body diagram serves the purpose of focusing attention on the action of the external forces, and therefore the diagram should not be cluttered with excessive extraneous information. Clearly distinguish force arrows from arrows representing quantities other than forces. For this purpose a colored pencil may be used.

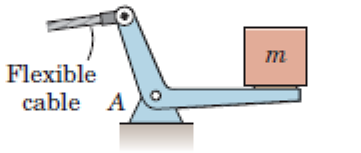
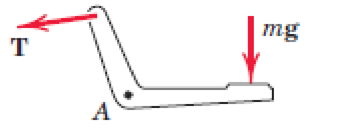
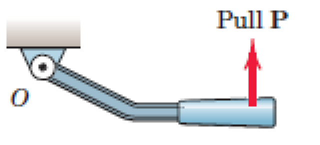
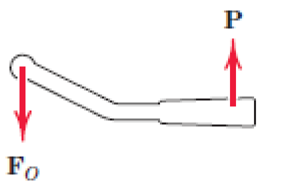
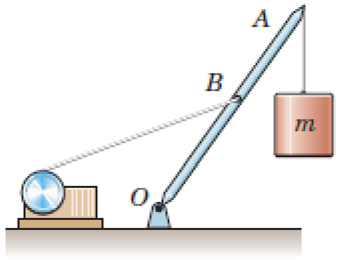
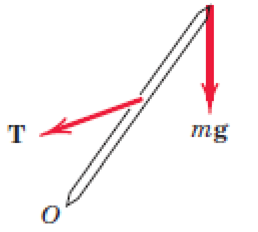
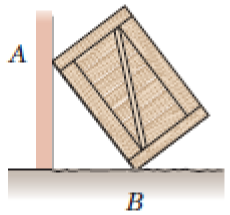
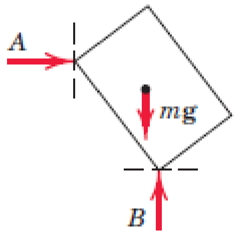
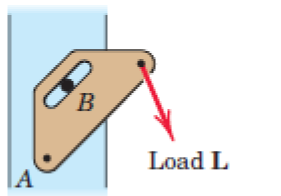
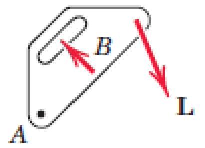
Examples of Free-Body Diagrams

Figure below gives four examples of mechanisms and structures together with their correct free-body diagrams. Dimensions and magnitudes are omitted for clarity. In each case we treat the entire system as

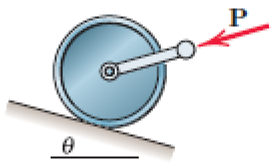
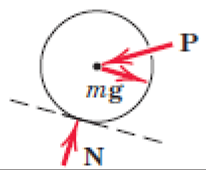
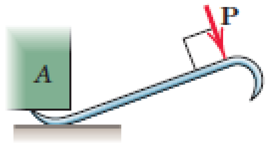
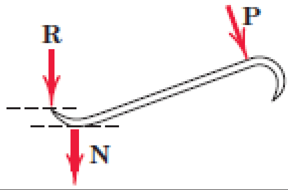
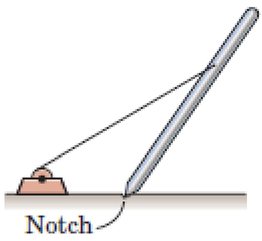
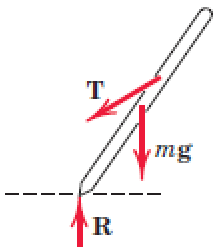
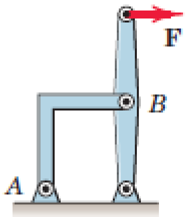
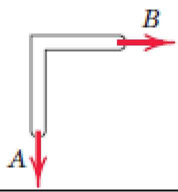
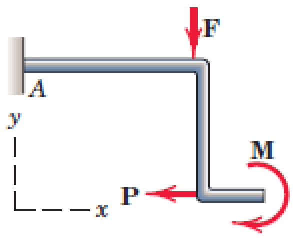
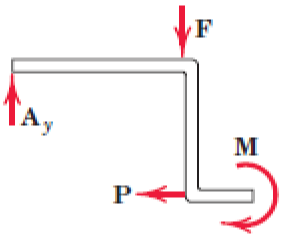


FREE-BODY DIAGRAM EXERCISES

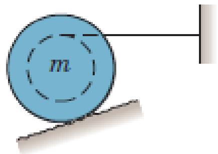
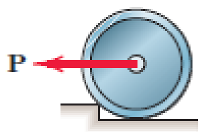
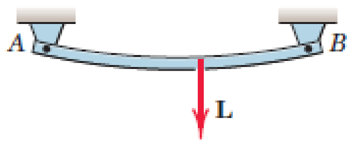
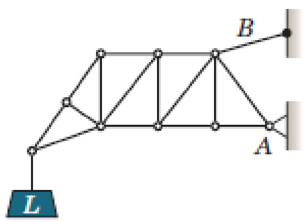
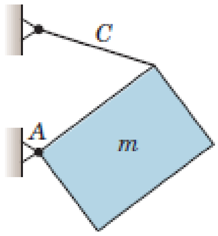
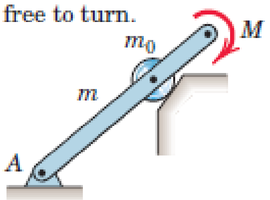
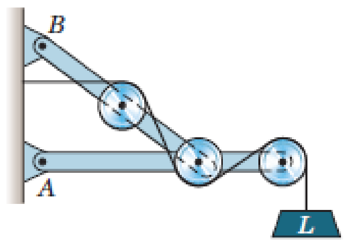
In each of the five following examples, the body to be isolated is shown in the left-hand diagram, and an incomplete free-body diagram (FBD) of the isolated body is shown on the right. Add whatever forces are necessary in each case to form a complete free-body diagram. The weights of the bodies are negligible unless otherwise indicated. Dimensions and numerical values are omitted for simplicity.

	Body	Incomplete FBD
1. Bell crank supporting mass m with pin support at A.		
2. Control lever applying torque to shaft at O.		
3. Boom OA, of negligible mass compared with mass m. Boom hinged at O and supported by hoisting cable at B.		
4. Uniform crate of mass m leaning against smooth vertical wall and supported on a rough horizontal surface.		
5. Loaded bracket supported by pin connection at A and fixed pin in smooth slot at B.		

In each of the five following examples, the body to be isolated is shown in the left-hand diagram, and either a wrong or an incomplete free-body diagram (FBD) is shown on the right. Make whatever changes or additions are necessary in each case to form a correct and complete free-body diagram. The weights of the bodies are negligible unless otherwise indicated. Dimensions and numerical values are omitted for simplicity

	Body	Wrong or Incomplete FBD
1. Lawn roller of mass m being pushed up incline θ.		
2. Prybar lifting body A having smooth horizontal surface. Bar rests on horizontal rough surface.		
3. Uniform pole of mass m being hoisted into position by winch. Horizontal supporting surface notched to prevent slipping of pole.		
4. Supporting angle bracket for frame; pin joints.		
5. Bent rod welded to support at A and subjected to two forces and couple.		

Draw a complete and correct free-body diagram of each of the bodies designated in the statements. The weights of the bodies are significant only if the mass is stated All forces, known and unknown, should be labeled. (Note: The sense of some reaction components cannot always be determined without numerical calculation.)

1. Uniform horizontal bar of mass m suspended by vertical cable at A and supported by rough inclined surface at B. 	5. Uniform grooved wheel of mass m supported by a rough surface and by action of horizontal cable. 
2. Wheel of mass m on verge of being rolled over curb by pull **P**. 	6. Bar, initially horizontal but deflected under load **L**. Pinned to rigid support at each end. 
3. Loaded truss supported by pin joint at A and by cable at B. 	7. Uniform heavy plate of mass m supported in vertical plane by cable C and hinge A. 
4. Uniform bar of mass m and roller of mass m_0 taken together. Subjected to couple M and supported as shown. Roller is free to turn. 	8. Entire frame, pulleys, and contacting cable to be isolated as a single unit. 

Equilibrium Conditions:-

we defined equilibrium as the condition in which the resultant of all forces and moments acting on a body is zero. Stated in another way, a body is in equilibrium if all forces and moments applied to it are in balance. These requirements are contained in the vector equations of equilibrium, , which in two dimensions may be written in scalar form as

$$\Sigma F_x = 0 \qquad \Sigma F_y = 0 \qquad \Sigma M_O = 0$$

The third equation represents the zero sum of the moments of all forces about any point O on or off the body. Equations above are the necessary and sufficient conditions for complete equilibrium in two dimensions. They are necessary conditions because, if they are not satisfied, there can be no force or moment balance. They are sufficient because once they are satisfied, there can be no imbalance, and equilibrium is assured.

These equations show that the acceleration of the mass center of a body is proportional to the resultant force ΣF acting on the body. Consequently, if a body moves with constant velocity (zero acceleration), the resultant force on it must be zero, and the body may be treated as in a state of translational equilibrium.

Categories of Equilibrium

Applications of equilibrium Eqs. fall naturally into a number of categories which are easily identified. The categories of force systems acting on bodies in two-dimensional equilibrium are summarized in Fig. and are explained further as follows.

Category 1, equilibrium of collinear forces, clearly requires only the one force equation in the direction of the forces (x-direction), since all other equations are automatically satisfied.

Category 2, equilibrium of forces which lie in a plane (x-y plane) and are concurrent at a point O, requires the two force equations only, since the moment sum about O, that is, about a z-axis through O, is necessarily zero. Included in this category is the case of the equilibrium of a particle.

Category 3, equilibrium of parallel forces in a plane, requires the one force equation in the direction of the forces (x-direction) and one moment equation about an axis (z-axis) normal to the plane of the forces.

Approach to Solving Problems

The sample problems at the end of this article illustrate the application of free-body diagrams and the equations of equilibrium to typical statics problems. These solutions should be studied thoroughly. In the problem work of this chapter and throughout mechanics, it is important to develop a logical and systematic approach which includes the following steps:

1. Identify clearly the quantities which are known and unknown.

2. Make an unambiguous choice of the body (or system of connected bodies treated as a single body) to be isolated and draw its complete free-body diagram, labeling all external known and unknown but identifiable forces and couples which act on it.

3. Choose a convenient set of reference axes, always using right handed axes when vector cross products are employed. Choose moment centers with a view to simplifying the calculations. Generally the best choice is one through which as many unknown forces pass as possible. Simultaneous solutions of equilibrium equations are frequently necessary, but can be minimized or avoided by a careful

choice of reference axes and moment centers.

4. Identify and state the applicable force and moment principles or equations which govern the equilibrium conditions of the problem. In the following sample problems these relations are shown in brackets and precede each major calculation.

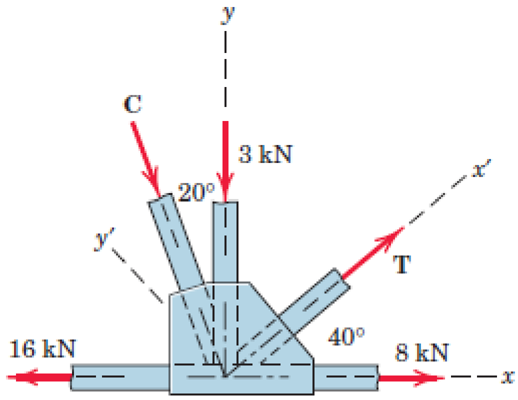
5. Match the number of independent equations with the number of unknowns in each problem.

6. Carry out the solution and check the results. In many problems engineering

judgment can be developed by first making a reasonable guess or estimate of the result prior to the calculation and then comparing the estimate with the calculated value.

SAMPLE PROBLEM

Ex1/ Determine the magnitudes of the forces C and T, which, along with the other three forces shown, act on the bridge-truss joint.



1 ***Solution.*** The given sketch constitutes the free-body diagram of the isolated section of the joint in question and shows the five forces which are in equilibrium.

Solution I (scalar algebra). For the x-y axes as shown we have

$$[\Sigma F_x = 0] \qquad 8 + T\cos 40° + C\sin 20° - 16 = 0$$

$$0.766T + 0.342C = 8 \qquad (a)$$

$$[\Sigma F_y = 0] \qquad T\sin 40° - C\cos 20° - 3 = 0$$

$$0.643T - 0.940C = 3 \qquad (b)$$

Simultaneous solution of Eqs. (a) and (b) produces

$$T = 9.09 \text{ kN} \qquad C = 3.03 \text{ kN}$$ *Ans.*

2 ***Solution II (scalar algebra).*** To avoid a simultaneous solution, we may use axes x'-y' with the first summation in the y'-direction to eliminate reference to T. Thus,

$$[\Sigma F_{y'} = 0] \qquad -C\cos 20° - 3\cos 40° - 8\sin 40° + 16\sin 40° = 0$$

$$C = 3.03 \text{ kN}$$ *Ans.*

$$[\Sigma F_{x'} = 0] \qquad T + 8\cos 40° - 16\cos 40° - 3\sin 40° - 3.03\sin 20° = 0$$

$$T = 9.09 \text{ kN}$$ *Ans.*

Ex2/ Calculate the tension T in the cable which supports the 1000-lb load with the pulley arrangement shown. Each pulley is free to rotate about its bearing, and the weights of all parts are small compared with the load. Find the magnitude of the total force on the bearing of pulley C.

Solution. The free-body diagram of each pulley is drawn in its relative position to the others. We begin with pulley *A*, which includes the only known force. With the unspecified pulley radius designated by *r*, the equilibrium of moments about its center *O* and the equilibrium of forces in the vertical direction require

1 $[\Sigma M_O = 0]$ $\quad T_1 r - T_2 r = 0 \quad T_1 = T_2$

$[\Sigma F_y = 0]$ $\quad T_1 + T_2 - 1000 = 0 \quad 2T_1 = 1000 \quad T_1 = T_2 = 500 \text{ lb}$

From the example of pulley *A* we may write the equilibrium of forces on pulley *B* by inspection as

$$T_3 = T_4 = T_2/2 = 250 \text{ lb}$$

For pulley *C* the angle $\theta = 30°$ in no way affects the moment of *T* about the center of the pulley, so that moment equilibrium requires

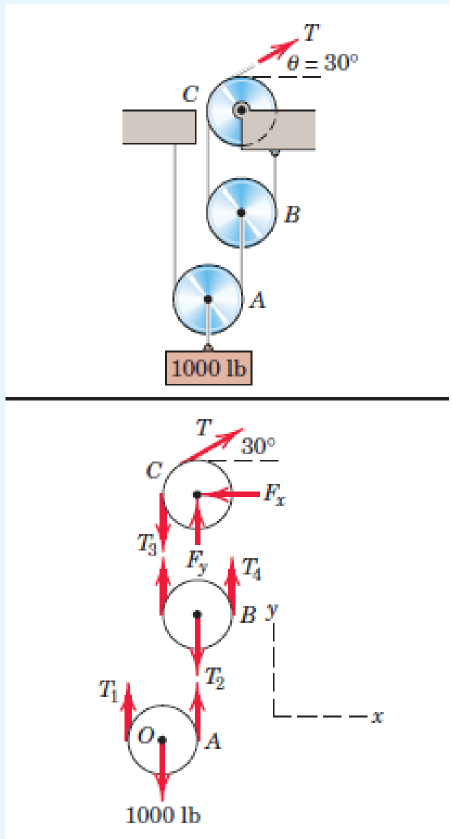
$$T = T_3 \quad \text{or} \quad T = 250 \text{ lb} \qquad \textit{Ans.}$$

Equilibrium of the pulley in the *x*- and *y*-directions requires

$[\Sigma F_x = 0]$ $\quad 250 \cos 30° - F_x = 0 \quad F_x = 217 \text{ lb}$

$[\Sigma F_y = 0]$ $\quad F_y + 250 \sin 30° - 250 = 0 \quad F_y = 125 \text{ lb}$

$[F = \sqrt{F_x^2 + F_y^2}]$ $\quad F = \sqrt{(217)^2 + (125)^2} = 250 \text{ lb} \qquad \textit{Ans.}$



Ex3/ The uniform 100-kg I-beam is supported initially by its end rollers on thehorizontal surface at A and B. By means of the cable at C it is desired to elevateend B to a position 3 m above end A. Determine the required tension P, the reaction at A, and the angle made by the beam with the horizontal in the elevated position.

Moment equilibrium about A eliminates force R and gives

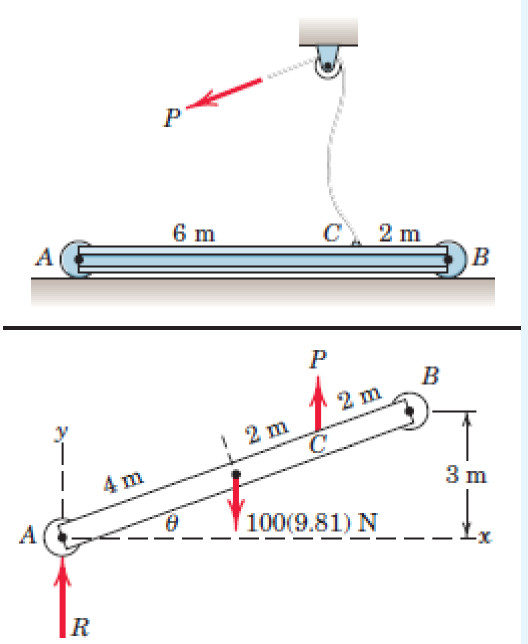
1 $[\Sigma M_A = 0]$ $\quad P(6\cos\theta) - 981(4\cos\theta) = 0 \qquad P = 654\text{ N}$

Equilibrium of vertical forces requires

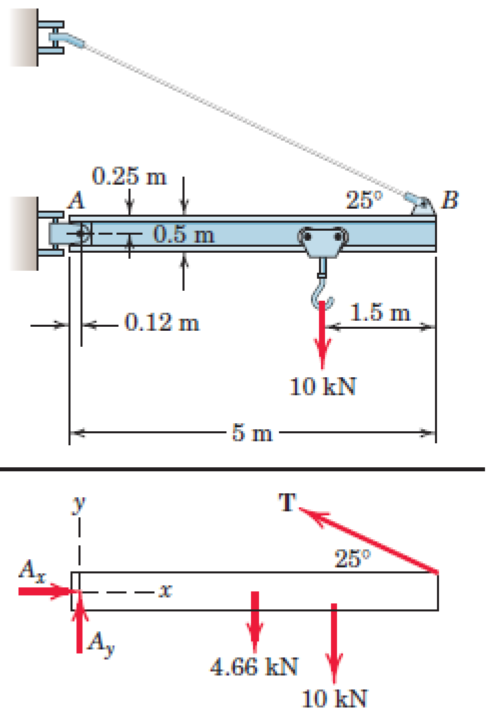
$[\Sigma F_y = 0]$ $\quad 654 + R - 981 = 0 \qquad R = 327\text{ N}$

The angle θ depends only on the specified geometry and is

$$\sin\theta = 3/8 \qquad \theta = 22.0°$$



Ex4/ Determine the magnitude T of the tension in the supporting cable and the magnitude of the force on the pin at A for the jib crane shown. The beam AB is a standard 0.5-m I-beam with a mass of 95 kg per meter of length.



Free-body diagram

2 $[\Sigma M_A = 0]$ $(T \cos 25°)0.25 + (T \sin 25°)(5 - 0.12) - 10(5 - 1.5 - 0.12) - 4.66(2.5 - 0.12) = 0$

from which $T = 19.61$ kN *Ans.*

Equating the sums of forces in the x- and y-directions to zero gives

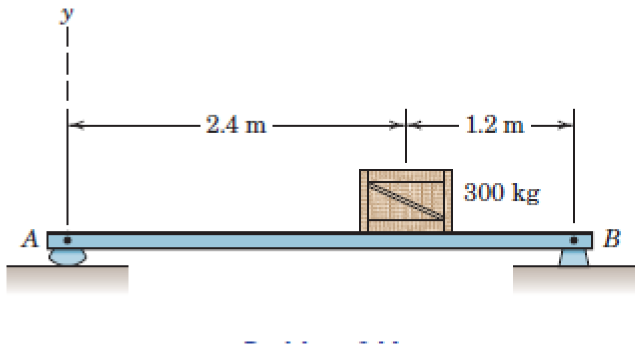
$[\Sigma F_x = 0]$ $A_x - 19.61 \cos 25° = 0$ $A_x = 17.77$ kN

$[\Sigma F_y = 0]$ $A_y + 19.61 \sin 25° - 4.66 - 10 = 0$ $A_y = 6.37$ kN

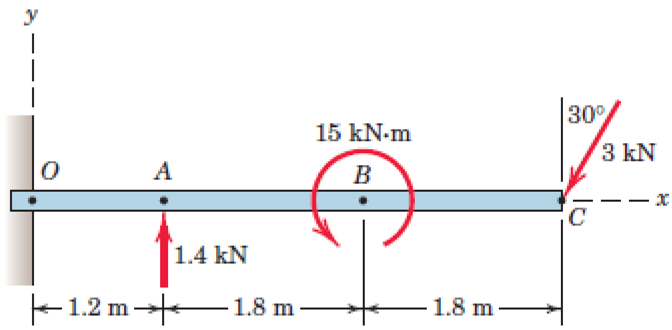
3 $[A = \sqrt{A_x^2 + A_y^2}]$ $A = \sqrt{(17.77)^2 + (6.37)^2} = 18.88$ kN *Ans.*

PROBLEMS

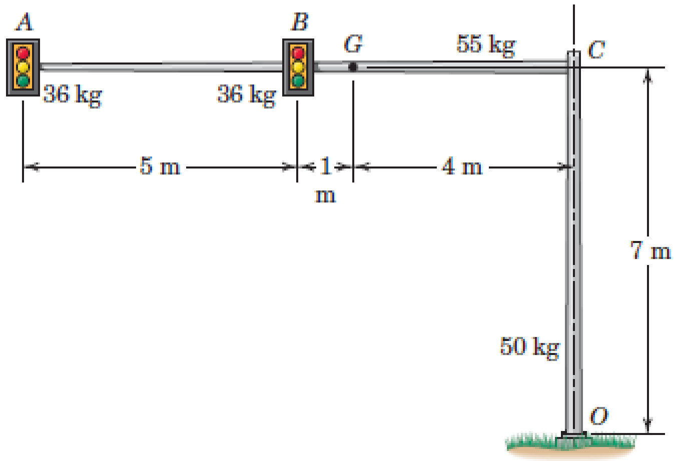
Q1/ The uniform beam has a mass of 50 kg per meter of length. Determine the reactions at the supports.



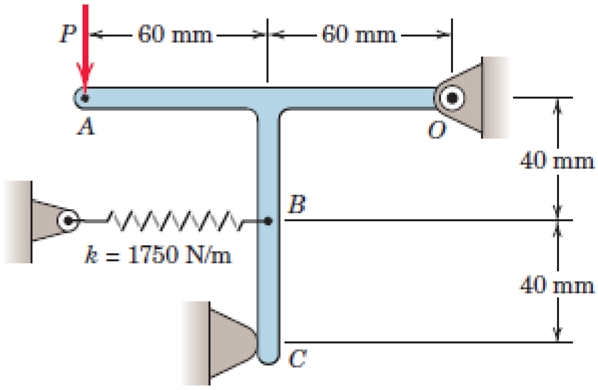
Q2/ The 500-kg uniform beam is subjected to the three external loads shown. Compute the reactions at the support point O. The x-y plane is vertical



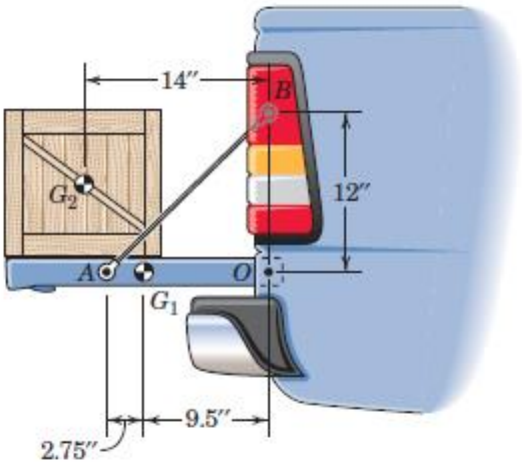
Q3 / Calculate the force and moment reactions at the bolted base O of the overhead traffic-signal assembly. Each traffic signal has a mass of 36 kg, while the masses of members OC and AC are 50 kg and 55 kg, respectively. The mass center of member AC is at G.



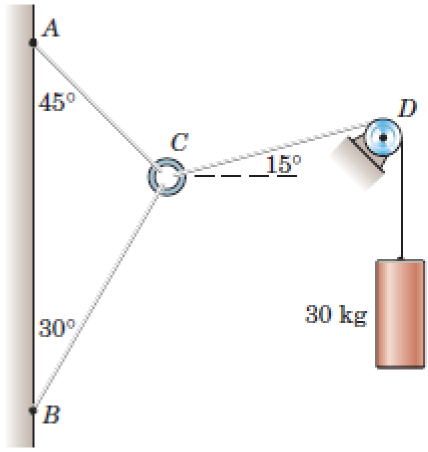
Q4/ When the 0.05-kg body is in the position shown, the linear spring is stretched 10 mm. Determine the force P required to break contact at C. Complete solutions for (a) including the effects of the weight and (b) neglecting the weight.



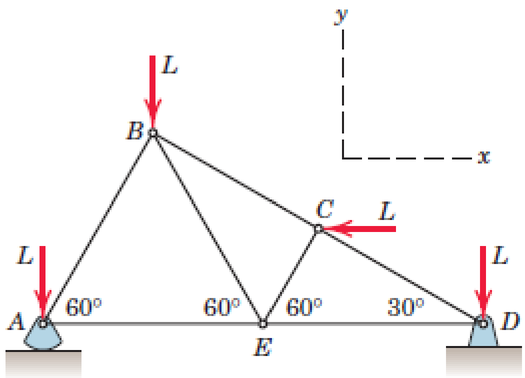
Q5/ A 120-lb crate rests on the 60-lb pickup tailgate. Calculate the tension T in each of the two restraining cables, one of which is shown. The centers of gravity are at G_1 and G_2. The crate is located midway between the two cables.



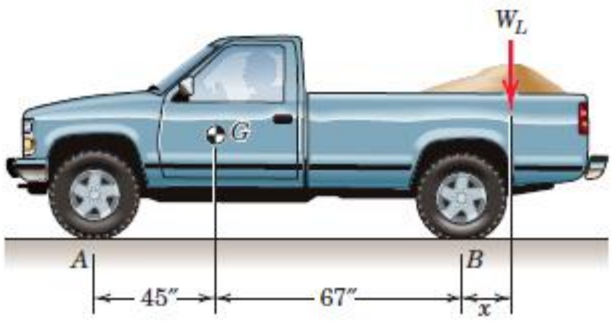
Q6/ Three cables are joined at the junction ring C. Determine the tensions in cables AC and BC caused by the weight of the 30-kg cylinder.



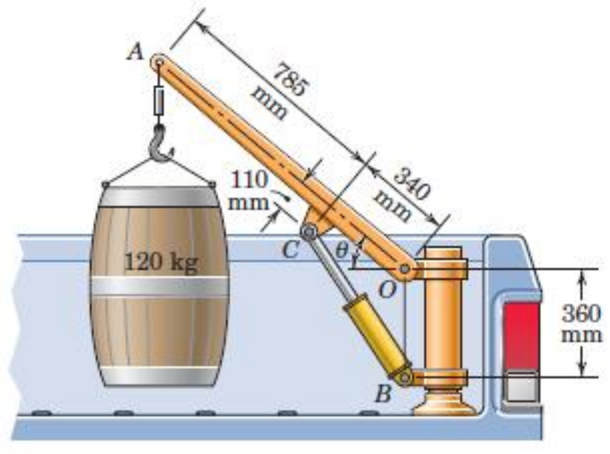
Q7/ The asymmetric simple truss is loaded as shown. Determine the reactions at A and D. Neglect the weight of the structure compared with the applied loads. Is knowledge of the size of the structure necessary?



Q8/ The indicated location of the center of gravity of the 3600-lb pickup truck is for the unladen condition. If X=16in a load whose center of gravity is in. behind the rear axle is added to the truck, determine the load weight W_L for which the normal forces under the front and rear wheels are equal



Q9/ The small crane is mounted on one side of the bed of a pickup truck. For the position θ=40° determine the magnitude of the force supported by the pin at O and the oil pressure p against the 50-mm-diameter piston of the hydraulic cylinder BC.



Structures

Introduction:-

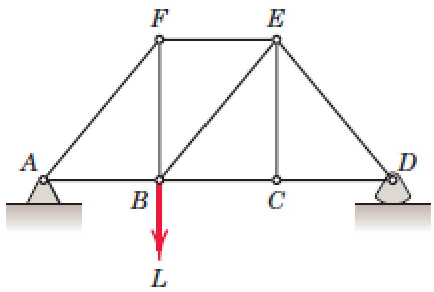
we studied the equilibrium of a single rigid body or a system of connected members treated as a single rigid body. We first drew a free-body diagram of the body showing all forces external to the isolated body and then we applied the force and moment equations of equilibrium.

we analyze the internal forces acting in several types of structures—namely, trusses, frames, and machines. In this treatment we consider only statically determinate structures, which do not have more supporting constraints than are necessary to maintain an equilibrium configuration. Thus, as we have already seen, the equations of equilibrium are adequate to determine all unknown reactions. The analysis of trusses, frames and machines, and beams under concentrated loads constitutes a straightforward application of the material developed in the previous two chapters. The basic procedure developed for isolating a body by constructing a correct free-body diagram is essential for the analysis of statically determinate

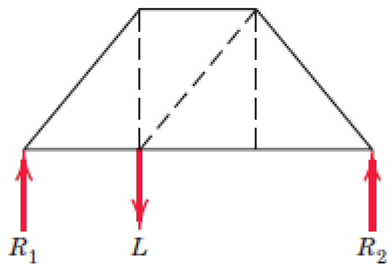
structures.

Truss Connections and Supports

When welded or riveted connections are used to join structural members, we may usually assume that the connection is a pin joint if the centerlines of the members are concurrent at the joint as in Fig.We also assume in the analysis of simple trusses that all external forces are applied at the pin connections. This condition is satisfied in most trusses. In bridge trusses the deck is usually laid on cross beams which are supported at the joints, For large trusses, a roller, rocker, or some kind of slip joint is used at one of the supports to provide for expansion and contraction due to temperature changes and for deformation from applied loads. Trusses and frames in which no such provision is made are statically indeterminate, Two methods for the force analysis of simple trusses will be given. Each method will be explained for the simple truss shown in Fig. a. The free-body diagram of the truss as a whole is shown in Fig. b external reactions are usually determined first, by applying the equilibrium equations to the truss as a whole. Then the force analysis of the remainder of the truss is performed.



(a)

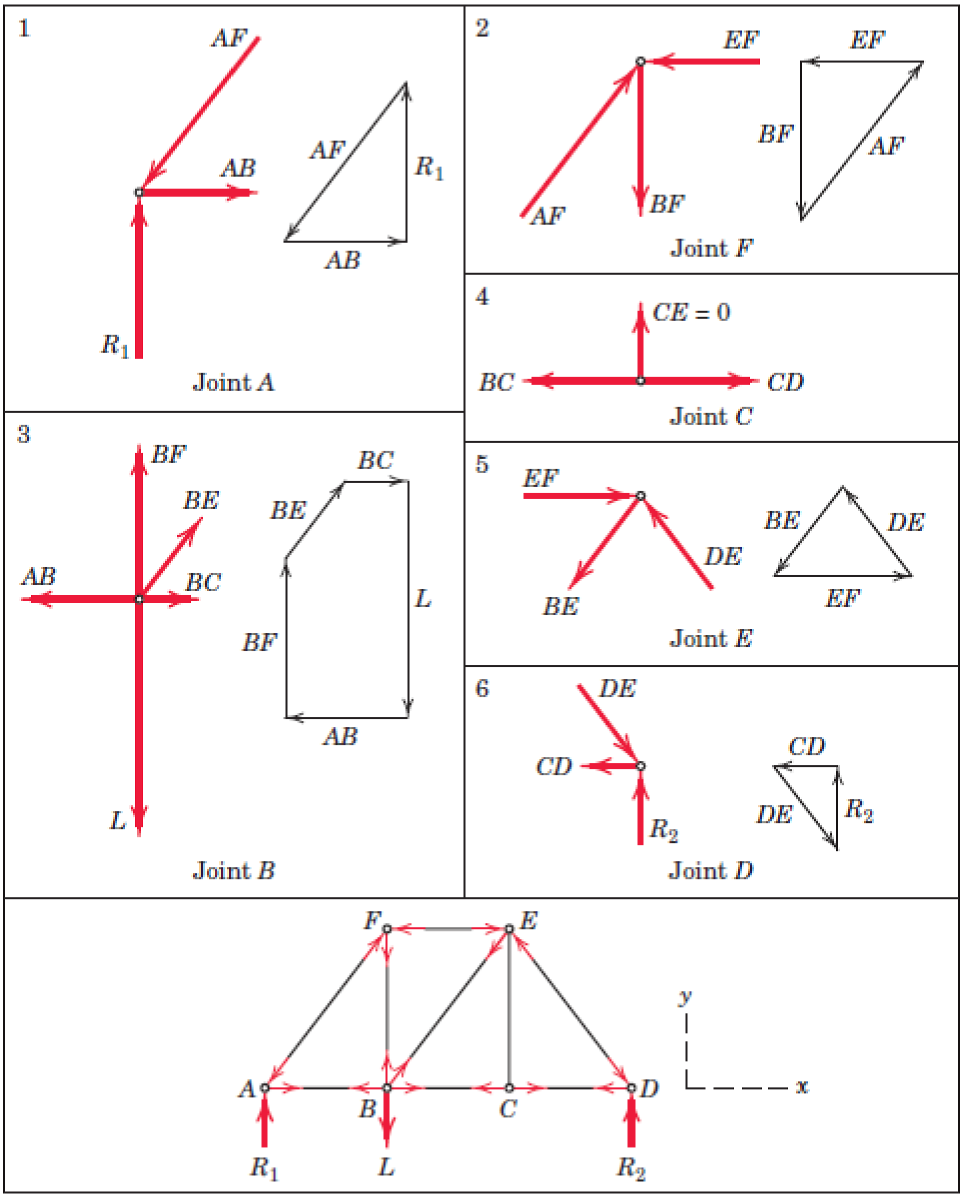


(b)

Method of Joints

This method for finding the forces in the members of a truss consists of satisfying the conditions of equilibrium for the forces acting on the connecting pin of each joint. The method therefore deals with the equilibrium of concurrent forces, and only two independent equilibrium equations are involved.

We begin the analysis with any joint where at least one known load exists and where not more than two unknown forces are present. The solution may be started with the pin at the left end. Its free-body diagramis shown in Fig. With the joints indicated by letters, we usually designate the force in each member by the two letters defining the ends of the member. The proper directions of the forces should be evident by inspection for this simple case. The free-body diagrams of portions of members AF and AB are also shown to clearly indicate the mechanism of the action and reaction. The member AB actually makes contact on the left side

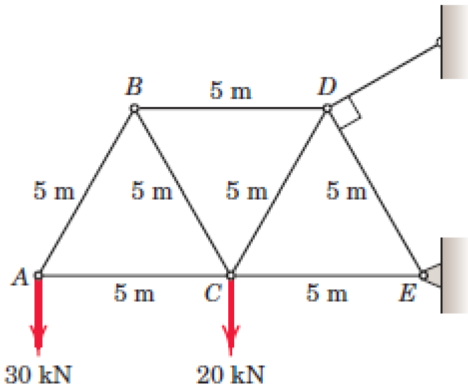


of the pin, although the force AB is drawn from the right side and is shown acting away from the pin. Thus, if we consistently draw the force arrows on the same side of the pin as the member, then tension (such as AB) will always be indicated by an arrow away from the pin, and compression (such as AF) will always be indicated by

an arrow toward the pin. The magnitude of AF is obtained from the equation ΣFy = 0 and AB is then found from ΣFx = 0.Joint F may be analyzed next, since it now contains only two unknowns, EF and BF. Proceeding to the next joint having no more than two unknowns, we subsequently analyze joints B, C, E, and D in that order. Figure shows the free-body diagram of each joint and its corresponding force polygon, which represents graphically the two equilibrium conditions ΣFx = 0 and ΣFy = 0. The numbers indicate the order in which the joints are analyzed. We note that, when joint D is finally reached, the computed reaction R2 must be in equilibrium with the forces in members CD and ED, which were determined previously from the two neighboring joints. This requirement provides a check on the correctness of our work. Note that isolation of joint C shows that the force in CE is zero when the equation ΣFy = 0 is applied. The force in this member would not be zero, of course, if an external vertical load were applied at C.

SAMPLE PROBLEM

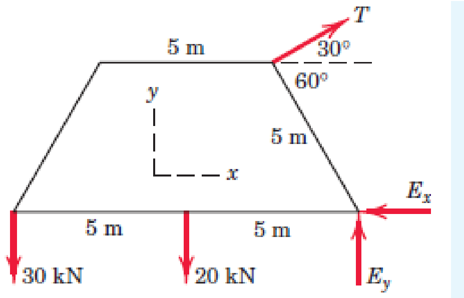
Compute the force in each member of the loaded cantilever truss by the method of joints.



Solution. If it were not desired to calculate the external reactions at *D* and *E*, the analysis for a cantilever truss could begin with the joint at the loaded end. However, this truss will be analyzed completely, so the first step will be to compute the external forces at *D* and *E* from the free-body diagram of the truss as a whole. The equations of equilibrium give

$[\Sigma M_E = 0]$	$5T - 20(5) - 30(10) = 0$	$T = 80$ kN
$[\Sigma F_x = 0]$	$80 \cos 30° - E_x = 0$	$E_x = 69.3$ kN
$[\Sigma F_y = 0]$	$80 \sin 30° + E_y - 20 - 30 = 0$	$E_y = 10$ kN

Next we draw free-body diagrams showing the forces acting on each of the connecting pins. The correctness of the assigned directions of the forces is verified when each joint is considered in sequence. There should be no question about the correct direction of the forces on joint A. Equilibrium requires

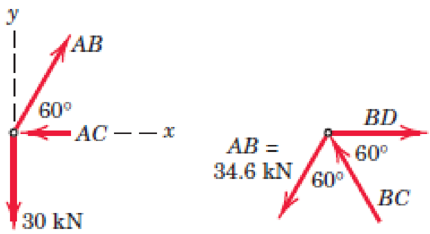


$[\Sigma F_y = 0]$ $\quad 0.866AB - 30 = 0 \quad AB = 34.6 \text{ kN } T$ *Ans.*

$[\Sigma F_x = 0]$ $\quad AC - 0.5(34.6) = 0 \quad AC = 17.32 \text{ kN } C$ *Ans.*

where T stands for tension and C stands for compression.

Joint B must be analyzed next, since there are more than two unknown forces on joint C. The force BC must provide an upward component, in which case BD must balance the force to the left. Again the forces are obtained from



Joint A Joint B

$[\Sigma F_y = 0]$ $\quad 0.866BC - 0.866(34.6) = 0 \quad BC = 34.6 \text{ kN } C$ *Ans.*

$[\Sigma F_x = 0]$ $\quad BD - 2(0.5)(34.6) = 0 \quad BD = 34.6 \text{ kN } T$ *Ans.*

Joint C now contains only two unknowns, and these are found in the same way as before:

$[\Sigma F_y = 0]$ $\quad 0.866CD - 0.866(34.6) - 20 = 0$

$CD = 57.7 \text{ kN } T$ *Ans.*

$[\Sigma F_x = 0]$ $\quad CE - 17.32 - 0.5(34.6) - 0.5(57.7) = 0$

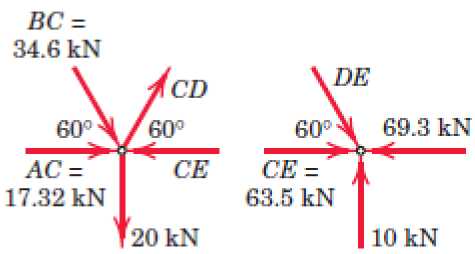
$CE = 63.5 \text{ kN } C$ *Ans.*

Finally, from joint E there results

$[\Sigma F_y = 0]$ $\quad 0.866DE = 10 \quad DE = 11.55 \text{ kN } C$ *Ans.*

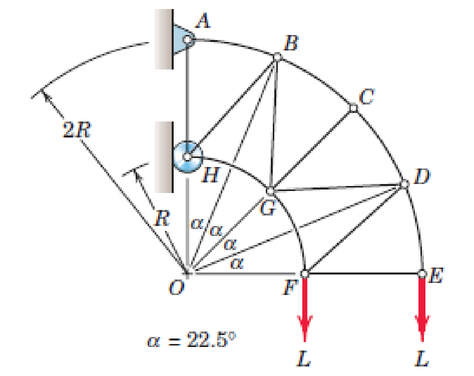
and the equation $\Sigma F_x = 0$ checks.

Note that the weights of the truss members have been neglected in comparison with the external loads.

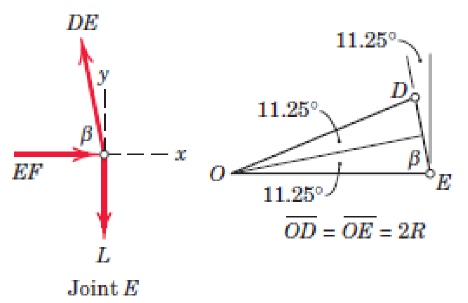


Joint C Joint E

SAMPLE PROBLEM /2
The simple truss shown supports the two loads, each of magnitude L. Determine the forces in members DE, DF, DG, and CD.



Solution. First of all, we note that the curved members of this simple truss are all two-force members, so that the effect of each curved member within the truss is the same as that of a straight member. We can begin with joint E because there are only two unknown member forces acting there. With reference to the free-body diagram and accompanying geometry for joint E, we note that



$$\beta = 180° - 11.25° - 90° = 78.8°.$$

$[\Sigma F_y = 0]$ $\quad DE \sin 78.8° - L = 0 \qquad DE = 1.020L\ T$ *Ans.*

$[\Sigma F_x = 0]$ $\quad EF - DE \cos 78.8° = 0 \qquad EF = 0.1989L\ C$

We must now move to joint F, as there are still three unknown members at joint D. From the geometric diagram,

$$\gamma = \tan^{-1}\left[\frac{2R \sin 22.5°}{2R \cos 22.5° - R}\right] = 42.1°$$

From the free-body diagram of joint F,

$[\Sigma F_x = 0]$ $\quad -GF \cos 67.5° + DF \cos 42.1° - 0.1989L = 0$

$[\Sigma F_y = 0]$ $\quad GF \sin 67.5° + DF \sin 42.1° - L = 0$

Simultaneous solution of these two equations yields

$$GF = 0.646L\ T \qquad DF = 0.601L\ T$$ *Ans.*

For member DG, we move to the free-body diagram of joint D and the accompanying geometry.

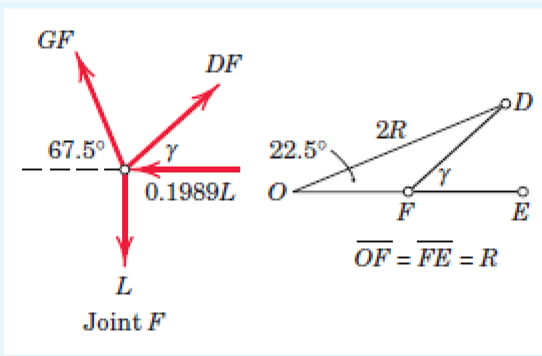
$$\delta = \tan^{-1}\left[\frac{2R \cos 22.5° - 2R \cos 45°}{2R \sin 45° - 2R \sin 22.5°}\right] = 33.8°$$

$$\epsilon = \tan^{-1}\left[\frac{2R \sin 22.5° - R \sin 45°}{2R \cos 22.5° - R \cos 45°}\right] = 2.92°$$

Then from joint D:

$[\Sigma F_x = 0]$ $\quad -DG \cos 2.92° - CD \sin 33.8° - 0.601L \sin 47.9° + 1.020L \cos 78.8° = 0$

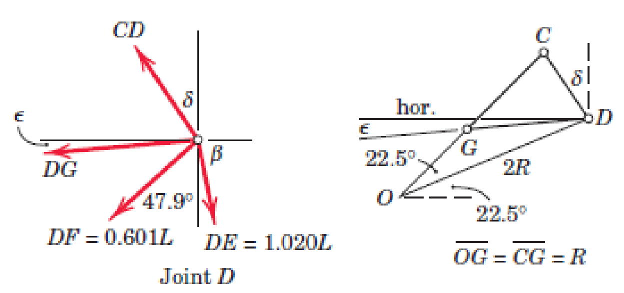
$[\Sigma F_y = 0]$ $\quad -DG \sin 2.92° + CD \cos 33.8° - 0.601L \cos 47.9° - 1.020L \sin 78.8° = 0$



The simultaneous solution is

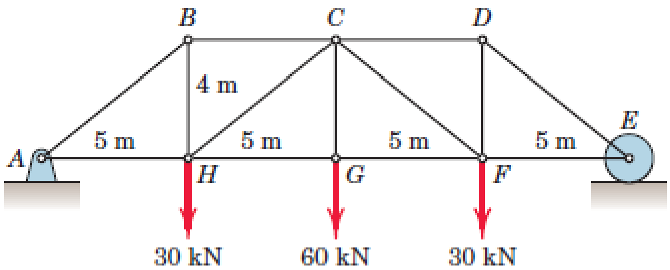
$CD = 1.617L\ T \qquad DG = -1.147L$ or $DG = 1.147L\ C$ *Ans.*

Note that ϵ is shown exaggerated in the accompanying figures.

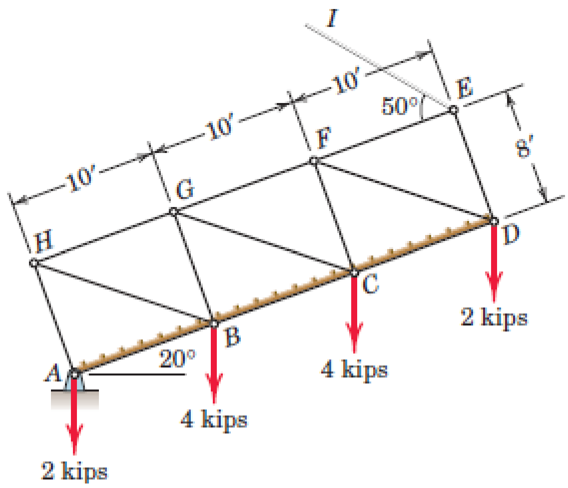


PROBLEMS

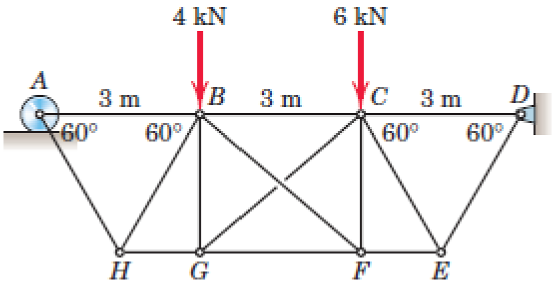
Q1/ Determine the force in each member of the loaded truss. Make use of the symmetry of the truss and of the loading.



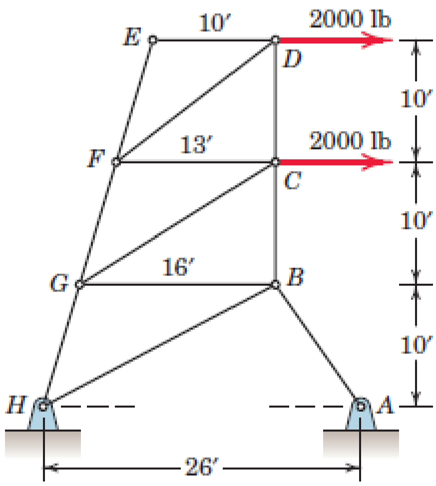
Q2/A drawbridge is being raised by a cable EI. The four joint loadings shown result from the weight of the roadway. Determine the forces in members EF, DE, DF, CD, and FG



Q3/ Calculate the forces in members AB, BH, and BG. Members BF and CG are cables which can support tension only.



Q4/ Calculate the forces in members CF, CG, and EF of the loaded truss

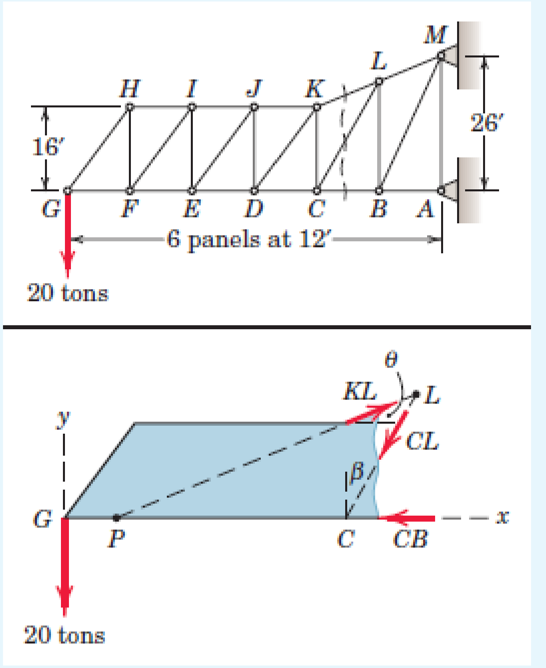


Method of Sections

When analyzing plane trusses by the method of joints, we need only two of the three equilibrium equations because the procedures involve concurrent forces at each joint. We can take advantage of the third or moment equation of equilibrium by selecting an entire section of the truss for the free body in equilibrium under the action of a no noncurrent system of forces. This method of sections has the basic advantage that the force in almost any desired member may be found directly from An analysis of a section which has cut that member. Thus, it is not necessary to proceed with the calculation from joint to joint until the member in question has been reached. In choosing a section of the truss, we note that, in general, not more than three members whose forces are unknown should be cut, since there are only three available independent equilibrium relations.

SAMPLE PROBLEM 1/

Calculate the forces induced in members KL, CL, and CB by the 20-ton load on the cantilever truss.



Solution. Although the vertical components of the reactions at A and M are statically indeterminate with the two fixed supports, all members other than AM are statically determinate. We may pass a section directly through members KL, CL, and CB and analyze the portion of the truss to the left of this section as a statically determinate rigid body.

2 Summing moments about L requires finding the moment arm $\overline{BL} = 16 + (26 - 16)/2 = 21$ ft. Thus,

$[\Sigma M_L = 0]$ $\quad 20(5)(12) - CB(21) = 0 \quad CB = 57.1 \text{ tons } C$ *Ans.*

Next we take moments about C, which requires a calculation of $\cos\theta$. From the given dimensions we see $\theta = \tan^{-1}(5/12)$ so that $\cos\theta = 12/13$. Therefore,

$[\Sigma M_C = 0]$ $\quad 20(4)(12) - \frac{12}{13}KL(16) = 0 \quad KL = 65 \text{ tons } T$ *Ans.*

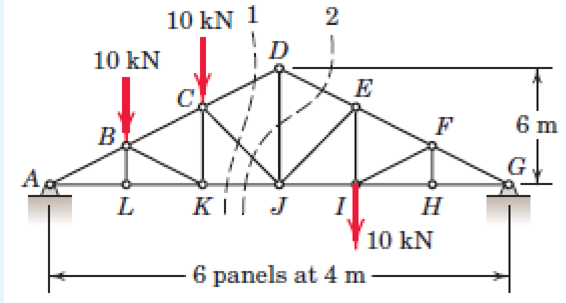
Finally, we may find CL by a moment sum about P, whose distance from C is given by $\overline{PC}/16 = 24/(26 - 16)$ or $\overline{PC} = 38.4$ ft. We also need β, which is given by $\beta = \tan^{-1}(\overline{CB}/\overline{BL}) = \tan^{-1}(12/21) = 29.7°$ and $\cos\beta = 0.868$. We now have

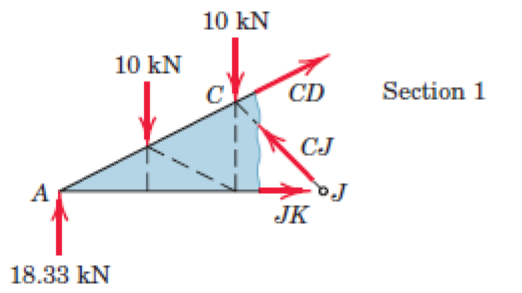
3 $[\Sigma M_P = 0]$ $\quad 20(48 - 38.4) - CL(0.868)(38.4) = 0$

$CL = 5.76 \text{ tons } C$ *Ans.*

SAMPLE PROBLEM /4

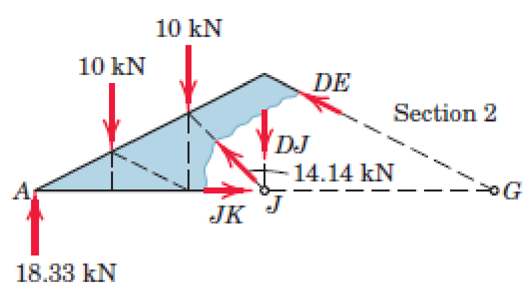
Calculate the force in member *DJ* of the Howe roof truss illustrated. Neglect any horizontal components of force at the supports.





Solution. It is not possible to pass a section through DJ without cutting four members whose forces are unknown. Although three of these cut by section 2 are concurrent at J and therefore the moment equation about J could be used to obtain DE, the force in DJ cannot be obtained from the remaining two equilibrium principles. It is necessary to consider first the adjacent section 1 before analyzing section 2.

The free-body diagram for section 1 is drawn and includes the reaction of 18.33 kN at A, which is previously calculated from the equilibrium of the truss as a whole. In assigning the proper directions for the forces acting on the three cut members, we see that a balance of moments about A eliminates the effects of CD and JK and clearly requires that CJ be up and to the left. A balance of moments about C eliminates the effect of the three forces concurrent at C and indicates that JK must be to the right to supply sufficient counterclockwise moment. Again it should be fairly obvious that the lower chord is under tension because of the bending tendency of the truss. Although it should also be apparent that the top chord is under compression, for purposes of illustration the force in CD will be arbitrarily assigned as tension.



Observe that a section through mem-

PROBLEMS

By the analysis of section 1, CJ is obtained from

$$[\Sigma M_A = 0] \qquad 0.707CJ(12) - 10(4) - 10(8) = 0 \qquad CJ = 14.14 \text{ kN } C$$

In this equation the moment of CJ is calculated by considering its horizontal and vertical components acting at point J. Equilibrium of moments about J requires

$$[\Sigma M_J = 0] \qquad 0.894CD(6) + 18.33(12) - 10(4) - 10(8) = 0$$

$$CD = -18.63 \text{ kN}$$

2 The moment of CD about J is calculated here by considering its two components as acting through D. The minus sign indicates that CD was assigned in the wrong direction.

Hence, $CD = 18.63 \text{ kN } C$

3 From the free-body diagram of section 2, which now includes the known value of CJ, a balance of moments about G is seen to eliminate DE and JK. Thus,

$$[\Sigma M_G = 0] \qquad 12DJ + 10(16) + 10(20) - 18.33(24) - 14.14(0.707)(12) = 0$$

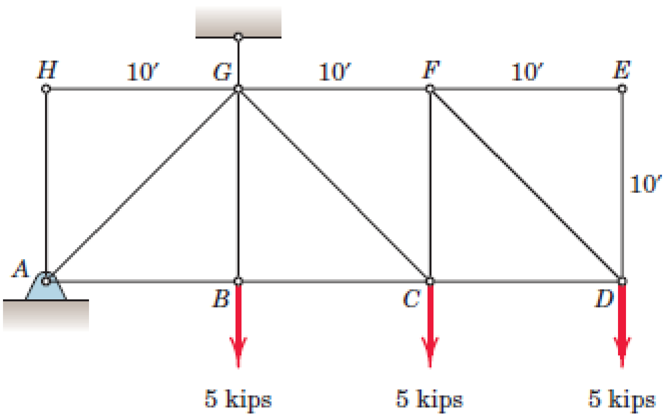
$$DJ = 16.67 \text{ kN } T \qquad \textit{Ans.}$$

Again the moment of CJ is determined from its components considered to be acting at J. The answer for DJ is positive, so that the assumed tensile direction is correct.

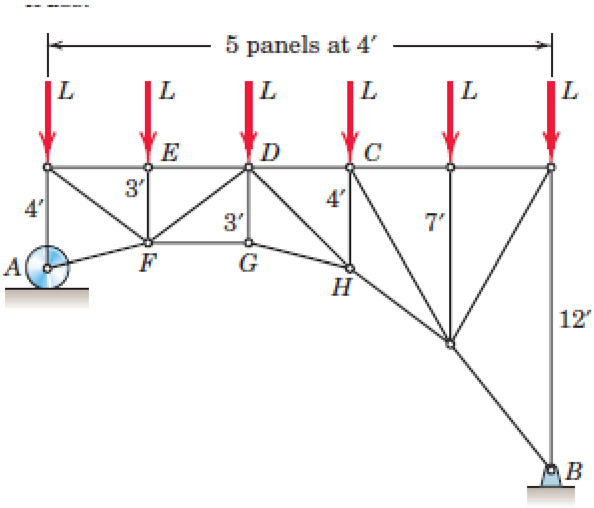
An alternative approach to the entire problem is to utilize section 1 to determine CD and then use the method of joints applied at D to determine DJ.

PROBLEMS

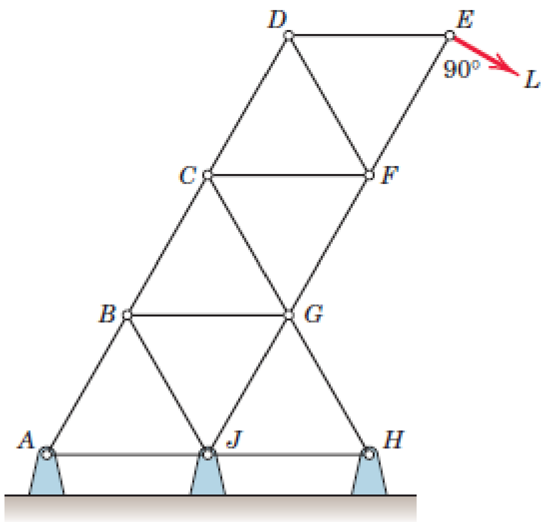
Q1/ Determine the force in member CG.



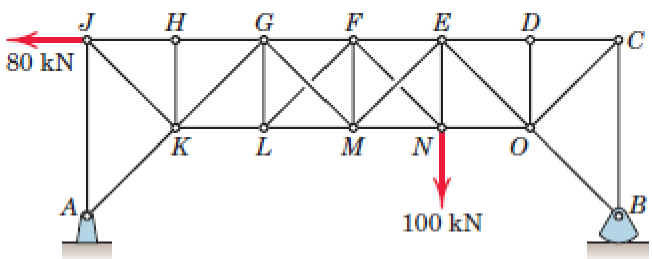
Q2/ Determine the force in member DG of the loaded truss.



Q3/ The truss is composed of equilateral triangles and supports the load L. Determine the forces in members CG and GF. Identify those members for which the equations of equilibrium are not sufficient to determine their forces



Q4/ The truss shown is composed of 45° right triangles. The crossed members in the center two panels are slender tie rods incapable of supporting compression. Retain the two rods which are under tension and compute the magnitudes of their tensions. Also find the force in member MN.



Friction

Introduction:-

action and reaction between contacting surfaces act normal to the surfaces. This assumption characterizes the interaction between smooth surfaces and was illustrated in Example 2 of Fig.. Although this ideal assumption often involves only a relatively small error, there are many problems in which we must consider the ability of contacting surfaces to support tangential as well as normal forces. Tangential forces generated between contacting surfaces are called friction forces and occur to some degree in the interaction between all real surfaces. Whenever a tendency exists for one contacting surface to slide along another surface, the friction forces developed are always in a direction to oppose this tendency. In some types of machines and processes we want to minimize the retarding effect of friction forces. Examples are bearings of all types, power screws, gears, the flow of fluids in pipes, and the propulsion of aircraft and missiles through the atmosphere. In other situations we wish to maximize the effects of friction, as in brakes, clutches, belt drives, and wedges. Wheeled vehicles depend on friction for both starting and stopping, and ordinary walking depends on friction between the shoe and the ground.

Types of Friction :-

In this article we briefly discuss the types of frictional resistance encountered in mechanics. The next article contains a more detailed account of the most common type of friction, dry friction.

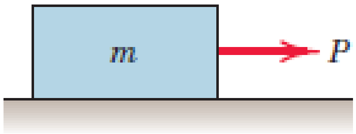
(a) Dry Friction. Dry friction occurs when the unlubricated surfaces of two solids are in contact under a condition of sliding or a tendency to slide. A friction force tangent to the surfaces of contact occurs both during the interval leading up to impending slippage and while slippage takes place. The direction of this friction force always opposes the motion or impending motion. This type of friction is also called *Coulomb friction*. The principles of dry or Coulomb friction were developed largely from the experiments of Coulomb in 1781 and from the work of Morin from 1831 to 1834. Although we do not yet have a comprehensive theory of dry friction, in Art. 6/3 we describe an analytical model sufficient to handle the vast majority of problems involving dry friction. This model forms the basis for most of this chapter.

(b) Fluid Friction. Fluid friction occurs when adjacent layers in a fluid (liquid or gas) are moving at different velocities. This motion causes frictional forces between fluid elements, and these forces depend on the relative velocity between layers. then there is no relative velocity, there is no fluid friction. Fluid friction depends not only on the velocity gradients within the fluid but also on the viscosity of the fluid, which is a measure of its resistance to shearing action between fluid layers. Fluid friction is treated in the study of fluid mechanics and will not be discussed further in this book.

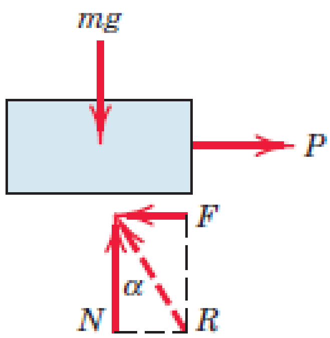
(c) Internal Friction. Internal friction occurs in all solid materials which are subjected to cyclical loading. For highly elastic materials the recovery from deformation occurs with very little loss of energy due to internal friction. For materials which have low limits of elasticity and which undergo appreciable plastic deformation during loading, a considerable amount of internal friction may accompany this deformation. The mechanism of internal friction is associated with the action of shear deformation, which is discussed in references on materials science. Because this book deals primarily with the external effects of forces, we will not discuss internal friction further.

Mechanism of Dry Friction :-

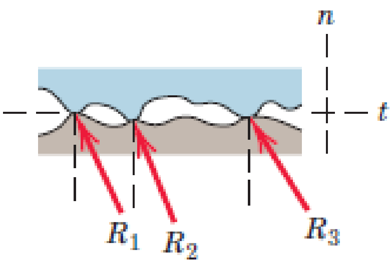
Consider a solid block of mass m resting on a horizontal surface, as shown in Fig. a. We assume that the contacting surfaces have some roughness. The experiment involves the application of a horizontal force P which continuously increases from zero to a value sufficient to move the block and give it an appreciable velocity. The free-body diagram of the block for any value of P is shown in Fig.b, where the tangential friction force exerted by the plane on the block is labeled F. This friction force acting on the body will always be in a direction to oppose motion or the tendency toward motion of the body. There is also a normal force N which in this case equals mg, and the total force R exerted by the supporting surface on the block is the resultant of N and F. A magnified view of the irregularities of the mating surfaces, Fig. c, helps us to visualize the mechanical action of friction. Support is necessarily intermittent and exists at the mating humps. The direction of each of the reactions on the block, R1, R2, R3, etc. depends not only on the geometric profile of the irregularities but also on the extent of local deformation at each contact point. The total normal force N is the sum



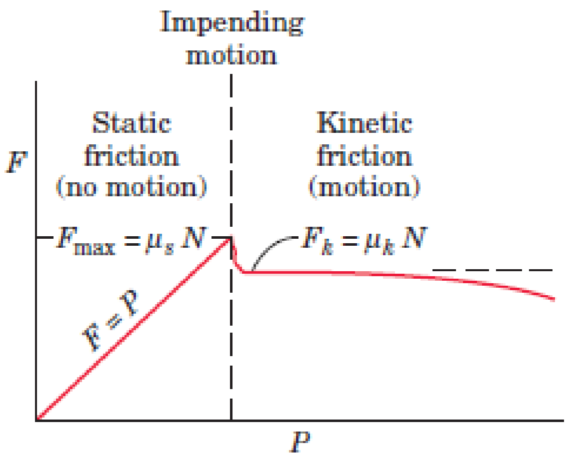
(a)



(b)



(c)



(d)

of the n-components of the R's, and the total frictional force F is the sum of the t-components of the R's. When the surfaces are in relative motion, the contacts are more nearly along the tops of the humps, and the t-components of the R's are smaller than when the surfaces are at rest relative to one another. This observation helps to explain the well known fact that the force P necessary to maintain motion is generally less than that required to start the block when the irregularities are more nearly in mesh. If we perform the experiment and record the friction force F as a function of P, we obtain the relation shown in Fig. 6/1d. When P is zero, equilibrium requires that there be no friction force. As P is increased, the friction force must be equal and opposite to P as long as the block does not slip. During this period the block is in equilibrium, and all forces acting on the block must satisfy the equilibrium equations. Finally, we reach a value of P which causes the block to slip and to move in the direction of the applied force. At this same time the friction force decreases slightly and abruptly. It then remains essentially constant for a time but then decreases still more as the velocity increases.

Static Friction:-

The region in Fig. d up to the point of slippage or impending motion is called the range of static friction, and in this range the value of the friction force is determined by the equations of equilibrium. This friction force may have any value from zero up to and including the maximum value. For a given pair of mating surfaces the experiment shows that this maximum value of static friction F_{max} is proportional to the normal force N. Thus, we may write

$$F_{\max} = \mu_s N$$

where μ_s is the proportionality constant, called the coefficient of static friction.

Be aware that Eq describes only the limiting or maximum value of the static friction force and not any lesser value. Thus, the equation applies only to cases where motion is impending with the friction force at its peak value. For a condition of static equilibrium when motion is not impending, the static friction force is

$$F < \mu_s N$$

Kinetic Friction

After slippage occurs, a condition of kinetic friction accompanies the ensuing motion. Kinetic friction force is usually somewhat less than the maximum static friction force. The kinetic friction force F_k is also proportional to the normal force. Thus

$$F_k = \mu_k N$$

Types of Friction Problems

We can now recognize the following three types of problems encountered in applications involving dry friction. The first step in solving a friction problem is to identify its type.

1. In the first type of problem, the condition of impending motion is known to exist. Here a body which is in equilibrium is on the verge of slipping, and the friction force equals the limiting static friction $F_{max} = \mu_s N$. The equations of equilibrium will, of course, also hold.

2. In the second type of problem, neither the condition of impending motion nor the condition of motion is known to exist. To determine the actual friction conditions, we first assume static equilibrium and then solve for the friction force F necessary for equilibrium. Three outcomes are possible:

(a) $F < (F_{max} = \mu_s N)$: Here the friction force necessary for equilibrium can be supported, and therefore the body is in static equilibrium as assumed. We emphasize that the actual friction force F is less than the limiting value F_{max} given by Eq ,and that F is determined solely by the equations of equilibrium.

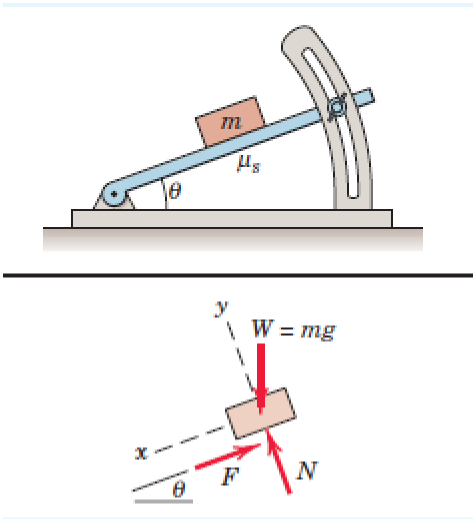
(b $F = (F_{max} = \mu_s N)$: Since the friction force F is at its maximum value Fmax, motion impends, as discussed in problem type (1). The assumption of static equilibrium is valid.

(c) $F > (F_{max} = \mu_s N)$: Clearly this condition is impossible, because the surfaces cannot support more force than the maximum μ_s N. The assumption of equilibrium is therefore invalid, and motion occurs. The friction force F is equal to μ_s N from Eq.

3. In the third type of problem, relative motion is known to exist between the contacting surfaces, and thus the kinetic coefficient of friction clearly applies. For this problem type, Eq always gives the kinetic friction force directly.

SAMPLE PROBLEM

1- / Determine the maximum angle **θ** which the adjustable incline may have with the horizontal before the block of mass m begins to slip. The coefficient of static friction between the block and the inclined surface is μ_s.



Solution. The free-body diagram of the block shows its weight $W = mg$, the normal force N, and the friction force F exerted by the incline on the block. The friction force acts in the direction to oppose the slipping which would occur if no friction were present.

Equilibrium in the x- and y-directions requires

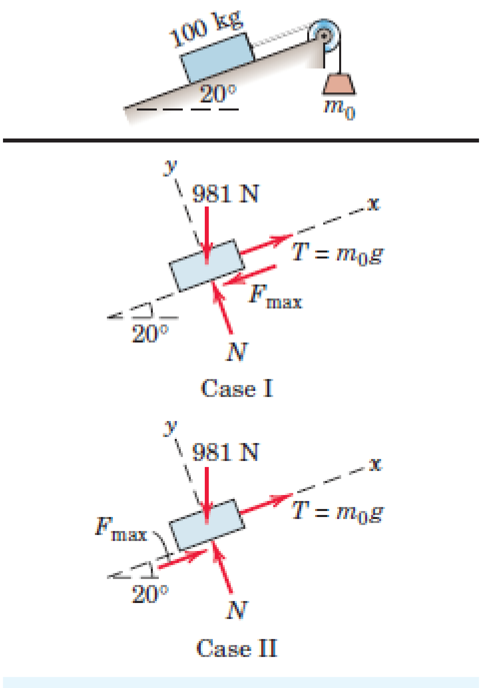
$$[\Sigma F_x = 0] \qquad mg \sin\theta - F = 0 \qquad F = mg \sin\theta$$

$$[\Sigma F_y = 0] \qquad -mg \cos\theta + N = 0 \qquad N = mg \cos\theta$$

Dividing the first equation by the second gives $F/N = \tan\theta$. Since the maximum angle occurs when $F = F_{max} = \mu_s N$, for impending motion we have

$$\mu_s = \tan\theta_{max} \qquad \text{or} \qquad \theta_{max} = \tan^{-1}\mu_s \qquad \textit{Ans.}$$

2-/ Determine the range of values which the mass m0 may have so that the 100-kg block shown in the figure will neither start moving up the plane nor slip down the plane. The coefficient of static friction for the contact surfaces is 0.30.



Solution. The maximum value of m0 will be given by the requirement for motion impending up the plane. The friction force on the block therefore acts down the plane, as shown in the free-body diagram of the block for Case I in the figure. With the weight mg = 100(9.81) = 981 N, the equations of equilibrium give

$[\Sigma F_y = 0]$ $N - 981 \cos 20° = 0$ $N = 922$ N

$[F_{max} = \mu_s N]$ $F_{max} = 0.30(922) = 277$ N

$[\Sigma F_x = 0]$ $m_0(9.81) - 277 - 981 \sin 20° = 0$ $m_0 = 62.4$ kg *Ans.*

The minimum value of m_0 is determined when motion is impending down the plane. The friction force on the block will act up the plane to oppose the tendency to move, as shown in the free-body diagram for Case II. Equilibrium in the x-direction requires

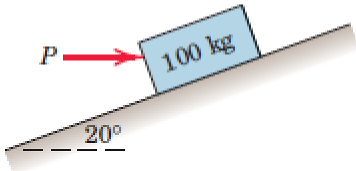
$[\Sigma F_x = 0]$ $m_0(9.81) + 277 - 981 \sin 20° = 0$ $m_0 = 6.01$ kg *Ans.*

Thus, m_0 may have any value from 6.01 to 62.4 kg, and the block will remain at rest.

In both cases equilibrium requires that the resultant of F_{max} and N be concurrent with the 981-N weight and the tension T.

SAMPLE PROBLEM /3

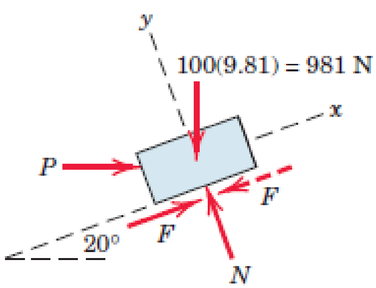
Determine the magnitude and direction of the friction force acting on the 100-kg block shown if, first, P = 500 N and, second, P = 100 N. The coefficient of static friction is 0.20, and the coefficient of kinetic friction is 0.17. The forces are applied with the block initially at rest



Solution. There is no way of telling from the statement of the problem whether the block will remain in equilibrium or whether it will begin to slip following the application of P. It is therefore necessary that we make an assumption, so we will take the friction force to be up the plane, as shown by the solid arrow. From the free-body diagram a balance of forces in both x- and y-directions gives

$[\Sigma F_x = 0]$ $P \cos 20° + F - 981 \sin 20° = 0$

$[\Sigma F_y = 0]$ $N - P \sin 20° - 981 \cos 20° = 0$



Case I. $P = 500$ N
Substitution into the first of the two equations gives

$$F = -134.3 \text{ N}$$

The negative sign tells us that *if* the block is in equilibrium, the friction force acting on it is in the direction opposite to that assumed and therefore is down the plane, as represented by the dashed arrow. We cannot reach a conclusion on the magnitude of F, however, until we verify that the surfaces are capable of supporting 134.3 N of friction force. This may be done by substituting $P = 500$ N into the second equation, which gives

$$N = 1093 \text{ N}$$

The maximum static friction force which the surfaces can support is then

$[F_{max} = \mu_s N]$ $\qquad F_{max} = 0.20(1093) = 219 \text{ N}$

Since this force is greater than that required for equilibrium, we conclude that the assumption of equilibrium was correct. The answer is, then,

$F = 134.3$ N down the plane *Ans.*

Case II. $P = 100$ N
Substitution into the two equilibrium equations gives

$$F = 242 \text{ N} \qquad N = 956 \text{ N}$$

But the maximum possible static friction force is

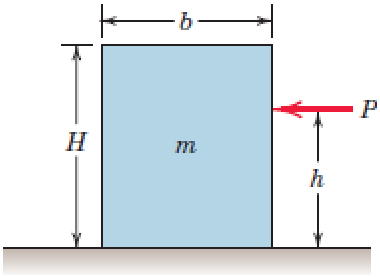
$[F_{max} = \mu_s N]$ $\qquad F_{max} = 0.20(956) = 191.2 \text{ N}$

(1) It follows that 242 N of friction cannot be supported. Therefore, equilibrium cannot exist, and we obtain the correct value of the friction force by using the kinetic coefficient of friction accompanying the motion down the plane. Hence, the answer is

$[F_k = \mu_k N]$ $\qquad F = 0.17(956) = 162.5$ N up the plane *Ans.*

SAMPLE PROBLEM /4

The homogeneous rectangular block of mass m, width b, and height H is placed on the horizontal surface and subjected to a horizontal force P which moves the block along the surface with a constant velocity. The coefficient of kinetic friction between the block and the surface is μ_k. Determine (a) the greatest value which h may have so that the block will slide without tipping over and (b) the location of a point C on the bottom face of the block through which the resultant of the friction and normal forces acts if h = H/2.



Solution. (*a*) With the block on the verge of tipping, we see that the entire reaction between the plane and the block will necessarily be at *A*. The free-body diagram of the block shows this condition. Since slipping occurs, the friction force is the limiting value $\mu_k N$, and the angle θ becomes $\theta = \tan^{-1}\mu_k$. The resultant of F_k and N passes through a point *B* through which *P* must also pass, since three coplanar forces in equilibrium are concurrent. Hence, from the geometry of the block

$$\tan\theta = \mu_k = \frac{b/2}{h} \qquad h = \frac{b}{2\mu_k} \qquad \textit{Ans.}$$

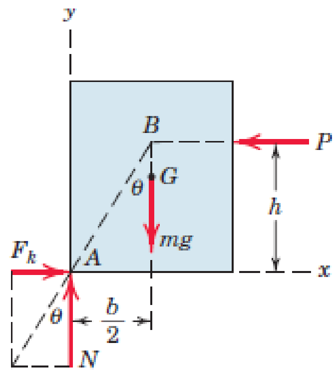
If *h* were greater than this value, moment equilibrium about *A* would not be satisfied, and the block would tip over.

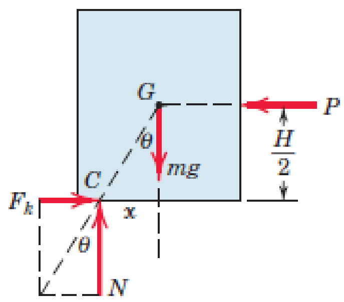
Alternatively, we may find *h* by combining the equilibrium requirements for the *x*- and *y*-directions with the moment-equilibrium equation about *A*. Thus,

$$[\Sigma F_y = 0] \qquad N - mg = 0 \qquad N = mg$$

$$[\Sigma F_x = 0] \qquad F_k - P = 0 \qquad P = F_k = \mu_k N = \mu_k mg$$

$$[\Sigma M_A = 0] \qquad Ph - mg\frac{b}{2} = 0 \qquad h = \frac{mgb}{2P} = \frac{mgb}{2\mu_k mg} = \frac{b}{2\mu_k} \qquad \textit{Ans.}$$





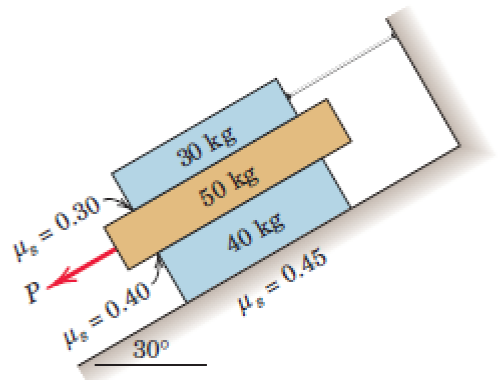
(*b*) With $h = H/2$ we see from the free-body diagram for case (*b*) that the resultant of F_k and N passes through a point *C* which is a distance *x* to the left of the vertical centerline through *G*. The angle θ is still $\theta = \phi = \tan^{-1}\mu_k$ as long as the block is slipping. Thus, from the geometry of the figure we have

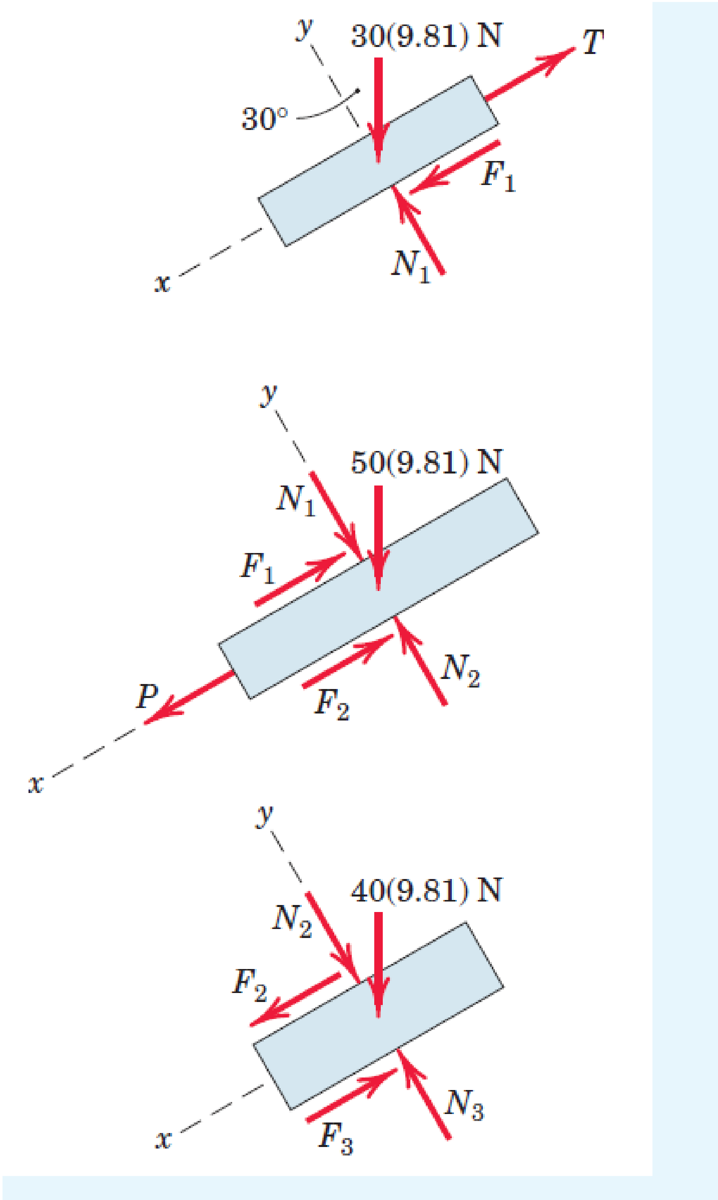
$$\frac{x}{H/2} = \tan\theta = \mu_k \qquad \text{so} \qquad x = \mu_k H/2 \qquad \textit{Ans.}$$

If we were to replace μ_k by the static coefficient μ_s, then our solutions would describe the conditions under which the block is (*a*) on the verge of tipping and (*b*) on the verge of slipping, both from a rest position.

SAMPLE PROBLEM /5

The three flat blocks are positioned on the 30° incline as shown, and a force P parallel to the incline is applied to the middle block. The upper block is prevented from moving by a wire which attaches it to the fixed support. The coefficient of static friction for each of the three pairs of mating surfaces is shown. Determine the maximum value which P may have before any slipping takes place





Solution. The free-body diagram of each block is drawn. The friction forces are assigned in the directions to oppose the relative motion which would occur if no friction were present. There are two possible conditions for impending motion. Either the 50-kg block slips and the 40-kg block remains in place, or the 50- and 40-kg blocks move together with slipping occurring between the 40-kg block and the incline.

The normal forces, which are in the *y*-direction, may be determined without reference to the friction forces, which are all in the *x*-direction. Thus,

$[\Sigma F_y = 0]$ (30-kg) $N_1 - 30(9.81)\cos 30° = 0$ $N_1 = 255$ N

(50-kg) $N_2 - 50(9.81)\cos 30° - 255 = 0$ $N_2 = 680$ N

(40-kg) $N_3 - 40(9.81)\cos 30° - 680 = 0$ $N_3 = 1019$ N

We will assume arbitrarily that only the 50-kg block slips, so that the 40-kg block remains in place. Thus, for impending slippage at both surfaces of the 50-kg block, we have

$[F_{max} = \mu_s N]$ $F_1 = 0.30(255) = 76.5$ N $F_2 = 0.40(680) = 272$ N

The assumed equilibrium of forces at impending motion for the 50-kg block gives

$[\Sigma F_x = 0]$ $P - 76.5 - 272 + 50(9.81)\sin 30° = 0$ $P = 103.1$ N

We now check on the validity of our initial assumption. For the 40-kg block with $F_2 = 272$ N the friction force F_3 would be given by

$[\Sigma F_x = 0]$ $272 + 40(9.81)\sin 30° - F_3 = 0$ $F_3 = 468$ N

But the maximum possible value of F_3 is $F_3 = \mu_s N_3 = 0.45(1019) = 459$ N. Thus, 468 N cannot be supported and our initial assumption was wrong. We conclude, therefore, that slipping occurs first between the 40-kg block and the incline. With the corrected value $F_3 = 459$ N, equilibrium of the 40-kg block for its impending motion requires

$[\Sigma F_x = 0]$ $F_2 + 40(9.81)\sin 30° - 459 = 0$ $F_2 = 263$ N

Equilibrium of the 50-kg block gives, finally,

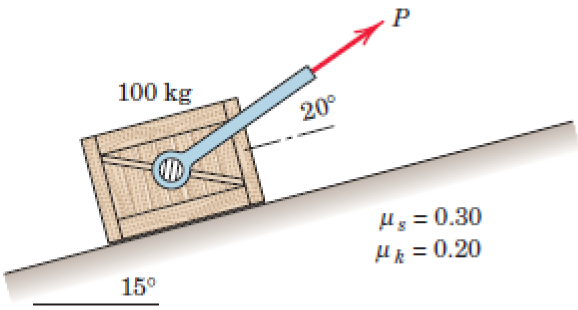
$[\Sigma F_x = 0]$ $P + 50(9.81)\sin 30° - 263 - 76.5 = 0$

$P = 93.8$ N *Ans.*

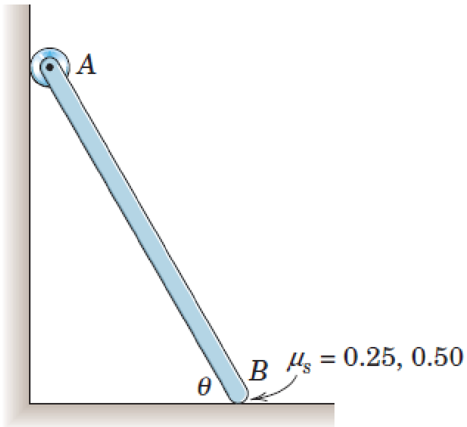
Thus, with $P = 93.8$ N, motion impends for the 50-kg and 40-kg blocks as a unit.

PROBLEMS

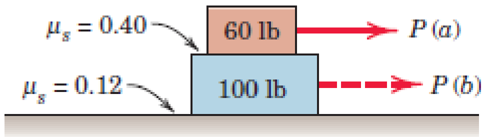
Q1/ The coefficients of static and kinetic friction between the 100-kg block and the inclined plane are 0.30 and 0.20, respectively. Determine (a) the friction force F acting on the block when P is applied with a magnitude of 200 N to the block at rest, (b) the force P required to initiate motion up the incline from rest, and (c) the friction force F acting on the block if P = 600 N.



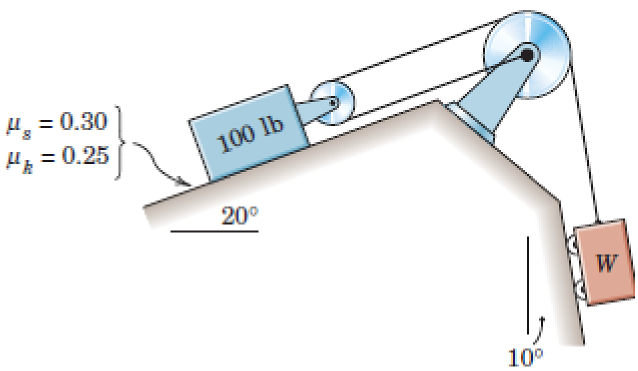
Q2/ The uniform slender bar has an ideal roller at its upper end A. If the coefficient of static friction at B is μ_s =0.25 , determine the minimum angle θ for which equilibrium is possible. Repeat for μ_s =0. 5.



Q3/ The force P is applied to (a) the 60-lb block and (b) the 100-lb block. For each case, determine the magnitude of P required to initiate motion.



Q4/ Determine the range of weights W for which the 100-lb block is in equilibrium. All wheels and pulleys have negligible friction

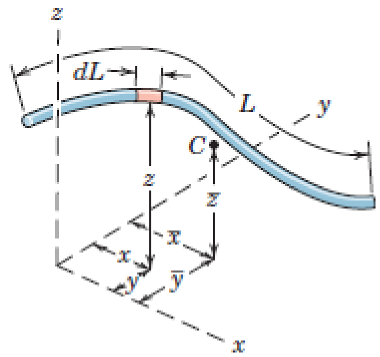


Centroids of Lines, Areas, and Volumes:-

When the density ρ of a body is uniform throughout, it will be a constant factor in both the numerators and denominators of Eqs. and will therefore cancel. The remaining expressions define a purely geometrical property of the body, since any reference to its mass properties has disappeared. The term centroid is used when the calculation concerns a geometrical shape only. When speaking of an actual physical body, we use the term center of mass. If the density is uniform throughout the body, the positions of the centroid and center of mass are identical, whereas if the density varies, these two points will, in general, not coincide. The calculation of centroids falls within three distinct categories, depending on whether we can model the shape of the body involved as a line, an area, or a volume.

(1) Lines. For a slender rod or wire of length L, cross-sectional area A, and density ρ, Fig., the body approximates a line segment, and dm = ρ A dL. If ρ and A are constant over the length of the rod, the coordinates of the center of mass also become the coordinates of the centroid C of the line segment, which, from Eqs., may be written

$$\bar{x} = \frac{\int x\,dL}{L} \qquad \bar{y} = \frac{\int y\,dL}{L} \qquad \bar{z} = \frac{\int z\,dL}{L}$$



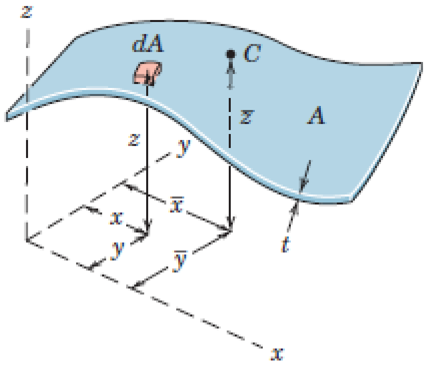
Note that, in general, the centroid C will not lie on the line. If the rod lies on a single plane, such as the x-y plane, only two coordinates need to be calculated.

(2) Areas. When a body of density has a small but constant thickness t, we can model it as a surface area A, Fig.. The mass of an element becomes dm = ρ t dA. Again, if and t are constant over the entire area, the coordinates of the center of

mass of the body also become the coordinates of the centroid C of the surface area, and from Eqs. the coordinates may be written

$$\bar{x} = \frac{\int x\,dA}{A} \qquad \bar{y} = \frac{\int y\,dA}{A} \qquad \bar{z} = \frac{\int z\,dA}{A}$$

The numerators in Eqsare called the first moments of area.* If the surface is curved, as illustrated in Fig. with the shell segment, all three coordinates will be involved. The centroid C for the curved surface will in general not lie on the surface. If the area is a flat surface in, say, the x-y plane, only the coordinates of C in that plane need to be calculated.



(3) Volumes. For a general body of volume V and density , the element has a mass dm= ρ dV. The density cancels if it is constant over the entire volume, and the coordinates of the center of mass also become the coordinates of the centroid C of the body. From Eqs. they become

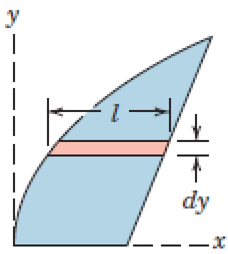
$$\bar{x} = \frac{\int x\,dV}{V} \qquad \bar{y} = \frac{\int y\,dV}{V} \qquad \bar{z} = \frac{\int z\,dV}{V}$$

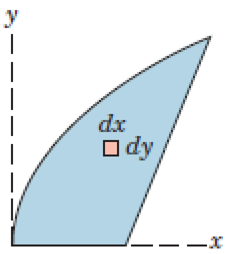
Choice of Element for Integration

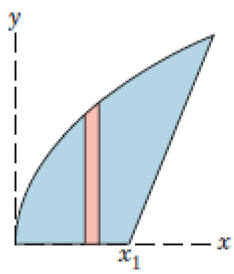
The principal difficulty with a theory often lies not in its concepts but in the procedures for applying it. With mass centers and centroids the concept of the moment principle is simple enough; the difficult steps are the choice of the differential element and setting up the integrals. The following five guidelines will be useful.

(1) Order of Element. Whenever possible, a first-order differential element should be selected in preference to a higher-order element so that only one integration will be required to cover the entire figure. Thus, in Fig. a first-order horizontal strip of area dA = l dy will require only one integration with respect to y to cover the entire figure. The second-order element dx dy will require two integrations, first with respect to x and second with respect to y, to cover the figure. As a further example, for the solid cone in Fig. we choose a first-order element in the form of a circular slice of volume $dV = \pi r^2 dy$. This choice requires only one integration, and thus is preferable to choosing a third-order element dV = dx dy dz, which would require three awkward integrations.

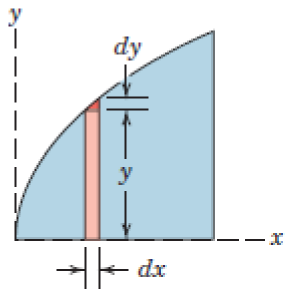
(2) Continuity. Whenever possible, we choose an element which can be integrated in one continuous operation to cover the figure. Thus, the horizontal strip in Fig. a would be preferable to the vertical strip in Fig., which, if used, would require two separate integrals because of the discontinuity in the expression for the height of the strip at $x = x_1$.



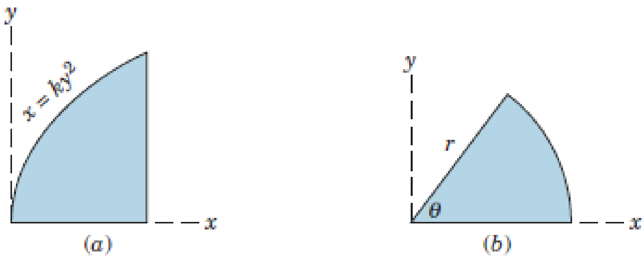




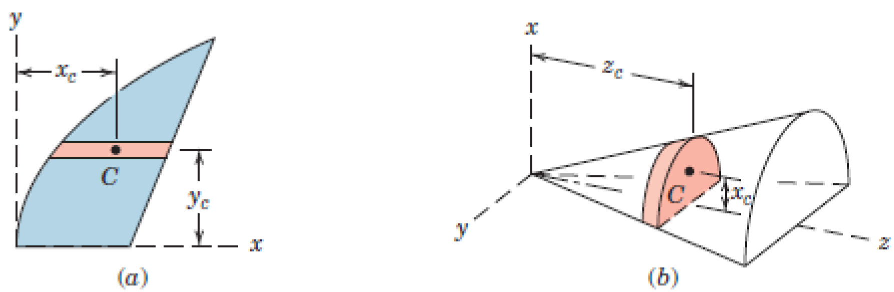
(3) Discarding Higher-Order Terms. Higher-order terms may always be dropped compared with lower-order terms (see Art. 1/7). Thus, the vertical strip of area under the curve in Fig. is given by the first-order term $dA = y\,dx$, and the second-order triangular area $\frac{1}{2}dx\,dy$ is discarded. In the limit, of course, there is no error.



(4) Choice of Coordinates. As a general rule, we choose the coordinate system which best matches the boundaries of the figure. Thus, the boundaries of the area in Fig. a are most easily described in rectangular coordinates, whereas the boundaries of the circular sector of Fig. b are best suited to polar coordinates.



(5) Centroidal Coordinate of Element. When a first- or secondorder differential element is chosen, it is essential to use the coordinate of the centroid of the element for the moment arm in expressing the moment of the differential element. Thus, for the horizontal strip of area in Fig. a, the moment of dA about the y-axis is x_c dA, where x_c is the x-coordinate of the centroid C of the element. Note that x_c is not the x which describes either boundary of the area. In the y-direction for this element the moment arm y_c of the centroid of the element is the same, in the limit, as the y-coordinates of the two boundaries. As a second example, consider the solid half-cone of Fig. b with the semicircular slice of differential thickness as the element of volume. The moment arm for the element in the x-direction is the distance x_c to the centroid of the face of the element and not the x-distance to the boundary of the element. On the other hand, in the z-direction the moment arm z_c of the centroid of the element is the same as the z-coordinate of the element. With these examples in mind, we rewrite in the form



$$\bar{x} = \frac{\int x_c \, dA}{A} \qquad \bar{y} = \frac{\int y_c \, dA}{A} \qquad \bar{z} = \frac{\int z_c \, dA}{A}$$

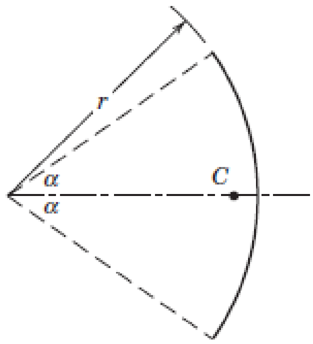
And

$$\bar{x} = \frac{\int x_c \, dV}{V} \qquad \bar{y} = \frac{\int y_c \, dV}{V} \qquad \bar{z} = \frac{\int z_c \, dV}{V}$$

It is essential to recognize that the subscript c serves as a reminder that the moment arms appearing in the numerators of the integral expressions for moments are always the coordinates of the centroids of the particular elements chosen.

SAMPLE PROBLEM /1

Centroid of a circular arc. Locate the centroid of a circular arc as shown in the figure



Solution. Choosing the axis of symmetry as the x-axis makes $\bar{y} = 0$. A differential element of arc has the length $dL = r\,d\theta$ expressed in polar coordinates, and the x-coordinate of the element is $r\cos\theta$.

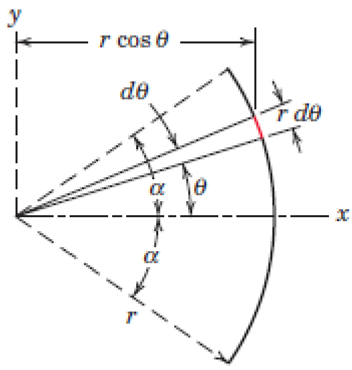
Applying the first of Eqs. 5/4 and substituting $L = 2\alpha r$ give

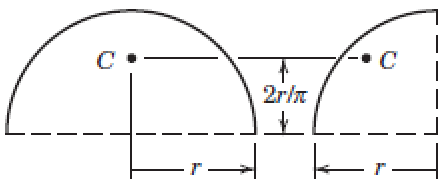
$$\left[L\bar{x} = \int x\,dL\right] \qquad (2\alpha r)\bar{x} = \int_{-\alpha}^{\alpha} (r\cos\theta)\,r\,d\theta$$

$$2\alpha r\bar{x} = 2r^2 \sin\alpha$$

$$\bar{x} = \frac{r\sin\alpha}{\alpha} \qquad \textit{Ans.}$$

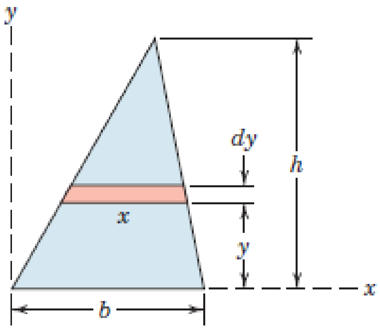
For a semicircular arc $2\alpha = \pi$, which gives $\bar{x} = 2r/\pi$. By symmetry we see immediately that this result also applies to the quarter-circular arc when the measurement is made as shown.





SAMPLE PROBLEM /2

Centroid of a triangular area. Determine the distance from the base of a triangle of altitude h to the centroid of its area.



Solution. The x-axis is taken to coincide with the base. A differential strip of area $dA = x\,dy$ is chosen. By similar triangles $x/(h-y) = b/h$. Applying the second of Eqs. 5/5a gives

1

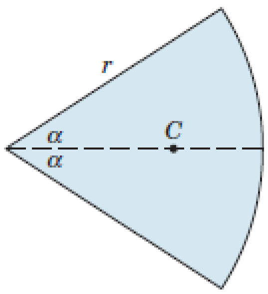
$$[A\bar{y} = \int y_c\, dA] \qquad \frac{bh}{2}\bar{y} = \int_0^h y\frac{b(h-y)}{h}\,dy = \frac{bh^2}{6}$$

and $$\bar{y} = \frac{h}{3}$$ *Ans.*

This same result holds with respect to either of the other two sides of the triangle considered a new base with corresponding new altitude. Thus, the centroid lies at the intersection of the medians, since the distance of this point from any side is one-third the altitude of the triangle with that side considered the base.

SAMPLE PROBLEM /3

Centroid of the area of a circular sector. Locate the centroid of the area of a circular sector with respect to its vertex.



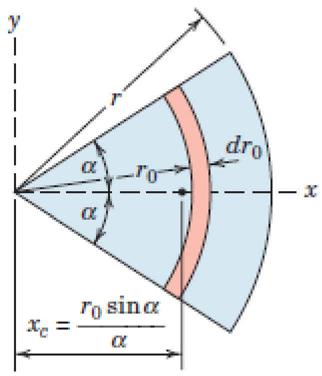
Solution I. The x-axis is chosen as the axis of symmetry, and $\bar{y}$ is therefore automatically zero. We may cover the area by moving an element in the form of a partial circular ring, as shown in the figure, from the center to the outer periphery. The radius of the ring is r_0 and its thickness is dr_0, so that its area is $dA = 2r_0\alpha\, dr_0$.

The x-coordinate to the centroid of the element from Sample Problem 5/1 is $x_c = r_0 \sin\alpha/\alpha$, where r_0 replaces r in the formula. Thus, the first of Eqs. 5/5a gives

$$[A\overline{x} = \int x_c\, dA] \qquad \frac{2\alpha}{2\pi}(\pi r^2)\overline{x} = \int_0^r \left(\frac{r_0 \sin\alpha}{\alpha}\right)(2r_0\alpha\, dr_0)$$

$$r^2\alpha\overline{x} = \frac{2}{3}r^3 \sin\alpha$$

$$\overline{x} = \frac{2}{3}\frac{r\sin\alpha}{\alpha} \qquad \textit{Ans.}$$



Solution I

Solution II. The area may also be covered by swinging a triangle of differential area about the vertex and through the total angle of the sector. This triangle, shown in the illustration, has an area $dA = (r/2)(r\,d\theta)$, where higher-order terms are neglected. From Sample Problem 5/2 the centroid of the triangular element of area is two-thirds of its altitude from its vertex, so that the x-coordinate to the centroid of the element is $x_c = \frac{2}{3}r\cos\theta$. Applying the first of Eqs. 5/5a gives

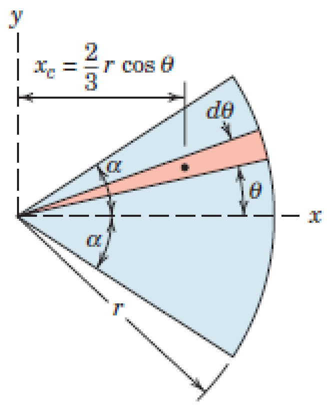
$$[A\overline{x} = \int x_c\, dA] \qquad (r^2\alpha)\overline{x} = \int_{-\alpha}^{\alpha} (\tfrac{2}{3}r\cos\theta)(\tfrac{1}{2}r^2\, d\theta)$$

$$r^2\alpha\overline{x} = \frac{2}{3}r^3\sin\alpha$$

and as before
$$\overline{x} = \frac{2}{3}\frac{r\sin\alpha}{\alpha} \qquad \textit{Ans.}$$

For a semicircular area $2\alpha = \pi$, which gives $\overline{x} = 4r/3\pi$. By symmetry we see immediately that this result also applies to the quarter-circular area where the measurement is made as shown.

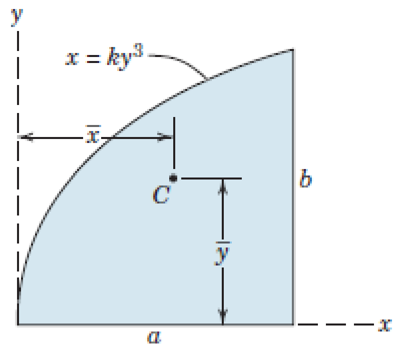
It should be noted that, if we had chosen a second-order element $r_0\, dr_0\, d\theta$, one integration with respect to θ would yield the ring with which *Solution I* began. On the other hand, integration with respect to r_0 initially would give the triangular element with which *Solution II* began.



Solution II

SAMPLE PROBLEM /4

Locate the centroid of the area under the curve $x = ky^3$ from $x = 0$ to $x = a$.

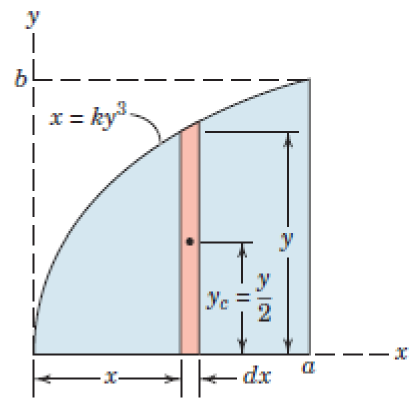


Solution I. A vertical element of area $dA = y\,dx$ is chosen as shown in the figure. The x-coordinate of the centroid is found from the first

$[A\bar{x} = \int x_c\, dA]$ $\quad \bar{x}\int_0^a y\, dx = \int_0^a xy\, dx$

Substituting $y = (x/k)^{1/3}$ and $k = a/b^3$ and integrating give

$$\frac{3ab}{4}\bar{x} = \frac{3a^2b}{7} \qquad \bar{x} = \tfrac{4}{7}a \qquad \textit{Ans}$$

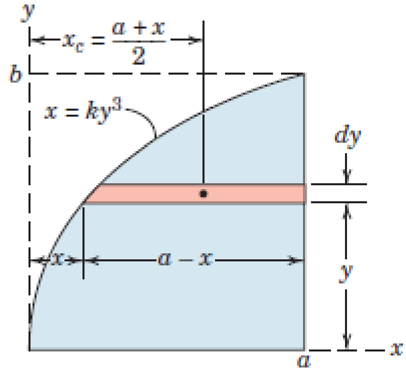


In the solution for $\bar{y}$ from the second of Eqs. 5/5a, the coordinate to the centroid of the rectangular element is $y_c = y/2$, where y is the height of the strip governed by the equation of the curve $x = ky^3$. Thus, the moment principle becomes

$[A\bar{y} = \int y_c\, dA]$ $\quad \frac{3ab}{4}\bar{y} = \int_0^a \left(\frac{y}{2}\right) y\, dx$

Substituting $y = b(x/a)^{1/3}$ and integrating give

$$\frac{3ab}{4}\bar{y} = \frac{3ab^2}{10} \qquad \bar{y} = \tfrac{2}{5}b \qquad \textit{Ans}$$



Solution II. The horizontal element of area shown in the lower figure may be employed in place of the vertical element. The x-coordinate to the centroid of the rectangular element is seen to be $x_c = x + \frac{1}{2}(a - x) = (a + x)/2$, which is simply the average of the coordinates a and x of the ends of the strip. Hence,

$[A\bar{x} = \int x_c\, dA]$ $\quad \bar{x}\int_0^b (a - x)\, dy = \int_0^b \left(\frac{a + x}{2}\right)(a - x)\, dy$

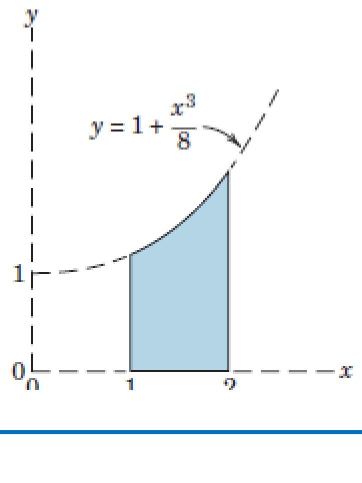
The value of $\bar{y}$ is found from

$[A\bar{y} = \int y_c\, dA]$ $\quad \bar{y}\int_0^b (a - x)\, dy = \int_0^b y(a - x)\, dy$

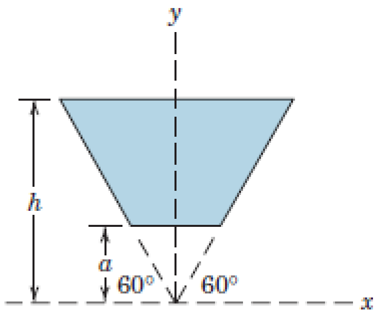
where $y_c = y$ for the horizontal strip. The evaluation of these integrals will check the previous results for $\bar{x}$ and $\bar{y}$.

PROBLEMS

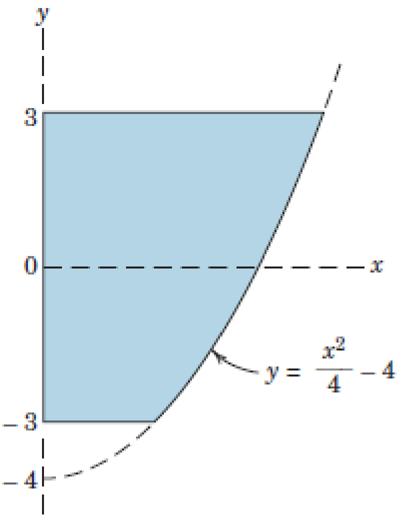
Q1/ Determine the x- and y-coordinates of the centroid of the shaded area.



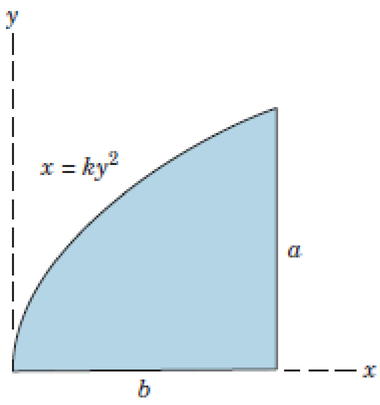
Q2/ Determine the y-coordinate of the centroid of the shaded area. Check your result for the special case (a=0)



Q3/ Locate the centroid of the shaded area shown.



Q4/ Determine the coordinates of the centroid of the shaded area.



Area Moments of Inertia

Introduction

When forces are distributed continuously over an area on which they act, it is often necessary to calculate the moment of these forces about some axis either in or perpendicular to the plane of the area. Frequently the intensity of the force (pressure or stress) is proportional to the distance of the line of action of the force from the moment axis. The elemental force acting on an element of area, then, is proportional to distance times differential area, and the elemental moment is proportional to distance squared times differential area. We see, therefore, that the total moment involves an integral of form $(\text{distance})^2$ d (area). This integral is called the moment of inertia or the second moment of the area. The integral is a function of the geometry of the area and occurs frequently in the applications of mechanics. Thus it is useful to develop its properties in some detail and to have these properties available for ready use when the integral arises.

Definitions

The following definitions form the basis for the analysis of area moments of inertia.

Rectangular and Polar Moments of Inertia Consider the area A in the x-y plane, Fig. A/2. The moments of inertia of the element dA about the x- and y-axes are, by definition, $dI_x = y^2\, dA$ and $dI_y = x^2\, dA$, respectively. The moments of inertia of A about the same axes are therefore

$$I_x = \int y^2\, dA$$

$$I_y = \int x^2\, dA$$

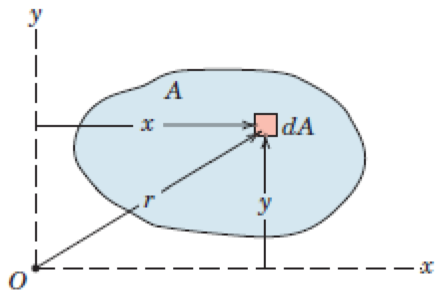


Figure A/2

The moment of inertia of dA about the pole O (z-axis) is, by similar definition, $dI_z = r^2\, dA$. The moment of inertia of the entire area about O is

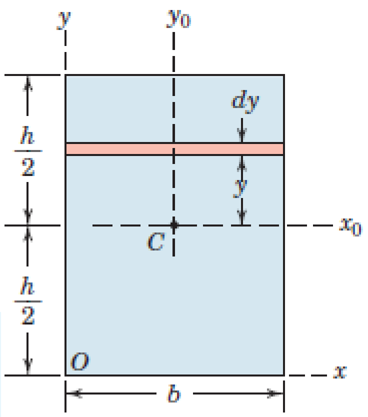
$$I_z = \int r^2\, dA$$

The expressions defined by . are called rectangular moments of inertia, whereas the expression of Eq. A/2 is called the polar moment of `inertia.* Because $x^2 + y^2 = r^2$, it is clear that

$$I_z = I_x + I_y$$

SAMPLE PROBLEM /1

Determine the moments of inertia of the rectangular area about the centroidal x_0- and y_0-axes, the centroidal polar axis z_0 through C, the x-axis, and the polar axis z through O.



Solution. For the calculation of the moment of inertia $\bar{I}_x$ about the x_0-axis, a horizontal strip of area $b\,dy$ is chosen so that all elements of the strip have the same y-coordinate. Thus,

$$\left[I_x = \int y^2\, dA\right] \qquad \bar{I}_x = \int_{-h/2}^{h/2} y^2 b\, dy = \tfrac{1}{12}bh^3 \qquad \textit{Ans.}$$

By interchange of symbols, the moment of inertia about the centroidal y_0-axis is

$$\bar{I}_y = \tfrac{1}{12}hb^3 \qquad \textit{Ans.}$$

The centroidal polar moment of inertia is

$$[\bar{I}_z = \bar{I}_x + \bar{I}_y] \qquad \bar{I}_z = \tfrac{1}{12}(bh^3 + hb^3) = \tfrac{1}{12}A(b^2 + h^2) \qquad \textit{Ans.}$$

By the parallel-axis theorem the moment of inertia about the x-axis is

$$[I_x = \bar{I}_x + Ad_x^{\,2}] \qquad I_x = \tfrac{1}{12}bh^3 + bh\left(\frac{h}{2}\right)^2 = \tfrac{1}{3}bh^3 = \tfrac{1}{3}Ah^2 \qquad \textit{Ans.}$$

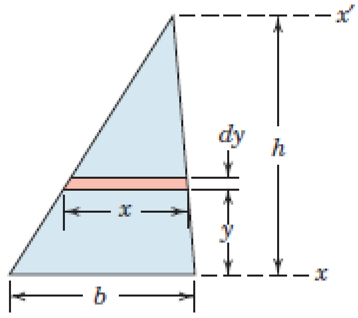
We also obtain the polar moment of inertia about O by the parallel-axis theorem, which gives us

$$[I_z = \bar{I}_z + Ad^2] \qquad I_z = \tfrac{1}{12}A(b^2 + h^2) + A\left[\left(\frac{b}{2}\right)^2 + \left(\frac{h}{2}\right)^2\right]$$

$$I_z = \tfrac{1}{3}A(b^2 + h^2) \qquad \textit{Ans.}$$

SAMPLE PROBLEM /2

Determine the moments of inertia of the triangular area about its base and about parallel axes through its centroid and vertex.



Solution. A strip of area parallel to the base is selected as shown in the figure, and it has the area $dA = x\,dy = [(h-y)b/h]\,dy$. By definition

$$[I_x = \int y^2\,dA] \qquad I_x = \int_0^h y^2 \frac{h-y}{h} b\,dy = b\left[\frac{y^3}{3} - \frac{y^4}{4h}\right]_0^h = \frac{bh^3}{12} \qquad \textit{Ans.}$$

By the parallel-axis theorem the moment of inertia $\bar{I}$ about an axis through the centroid, a distance $h/3$ above the x-axis, is

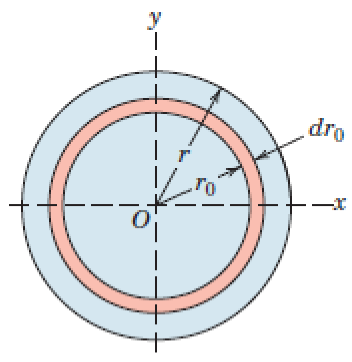
$$[\bar{I} = I - Ad^2] \qquad \bar{I} = \frac{bh^3}{12} - \left(\frac{bh}{2}\right)\left(\frac{h}{3}\right)^2 = \frac{bh^3}{36} \qquad \textit{Ans.}$$

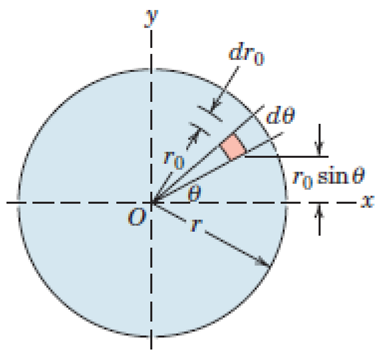
A transfer from the centroidal axis to the x'-axis through the vertex gives

$$[I = \bar{I} + Ad^2] \qquad I_{x'} = \frac{bh^3}{36} + \left(\frac{bh}{2}\right)\left(\frac{2h}{3}\right)^2 = \frac{bh^3}{4} \qquad \textit{Ans.}$$

SAMPLE PROBLEM /3

Calculate the moments of inertia of the area of a circle about a diametral axis and about the polar axis through the center. Specify the radii of gyration.





Solution. A differential element of area in the form of a circular ring may be used for the calculation of the moment of inertia about the polar z-axis through O since all elements of the ring are equidistant from O. The elemental area is $dA = 2\pi r_0\, dr_0$, and thus,

$$\left[I_z = \int r^2\, dA\right] \qquad I_z = \int_0^r r_0^{\,2}(2\pi r_0\, dr_0) = \frac{\pi r^4}{2} = \frac{1}{2}Ar^2 \qquad \textit{Ans.}$$

The polar radius of gyration is

$$\left[k = \sqrt{\frac{I}{A}}\right] \qquad k_z = \frac{r}{\sqrt{2}} \qquad \textit{Ans.}$$

By symmetry $I_x = I_y$, so that from Eq. A/3

$$[I_z = I_x + I_y] \qquad I_x = \frac{1}{2}I_z = \frac{\pi r^4}{4} = \frac{1}{4}Ar^2 \qquad \textit{Ans.}$$

The radius of gyration about the diametral axis is

$$\left[k = \sqrt{\frac{I}{A}}\right] \qquad k_x = \frac{r}{2} \qquad \textit{Ans.}$$

The foregoing determination of I_x is the simplest possible. The result may also be obtained by direct integration, using the element of area $dA = r_0\, dr_0\, d\theta$ shown in the lower figure. By definition

$$\left[I_x = \int y^2\, dA\right] \qquad I_x = \int_0^{2\pi}\int_0^r (r_0 \sin\theta)^2 r_0\, dr_0\, d\theta$$

$$= \int_0^{2\pi} \frac{r^4 \sin^2\theta}{4}\, d\theta$$

$$= \frac{r^4}{4}\frac{1}{2}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{2\pi} = \frac{\pi r^4}{4} \qquad \textit{Ans.}$$

Composite Areas

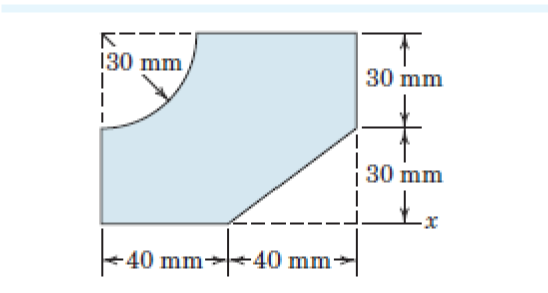
It is frequently necessary to calculate the moment of inertia of an area composed of a number of distinct parts of simple and calculable geometric shape. Because a moment of inertia is the integral or sum of the products of distance squared times element of area, it follows that the moment of inertia of a positive area is always a positive quantity. The moment of inertia of a composite area about a particular axis is therefore simply the sum of the moments of inertia of its component parts about the same axis. It is often convenient to regard a composite area as being composed of positive and negative parts. We may then treat the moment of inertia of a negative area as a negative quantity.

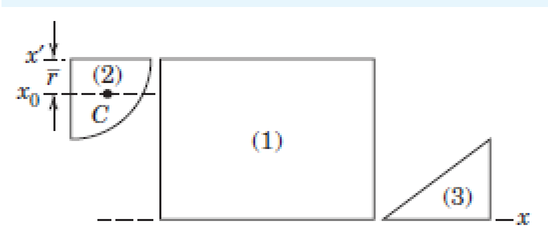
When a composite area is composed of a large number of parts, it is convenient to tabulate the results for each of the parts in terms of its area A, its centroidal moment of inertia $\bar{I}$, the distance d from its centroidal axis to the axis about which the moment of inertia of the entire section is being computed, and the product Ad^2. For any one of the parts the moment of inertia about the desired axis by the transfer-of-axis theorem is $\bar{I} + Ad^2$. Thus, for the entire section the desired moment of inertia becomes $I = \Sigma\bar{I} + \Sigma Ad^2$.

For such an area in the x-y plane, for example, and with the notation of Fig. A/4, where $\bar{I}_x$ is the same as I_{x_0} and $\bar{I}_y$ is the same as I_{y_0} the tabulation would include

SAMPLE PROBLEM /4

Calculate the moment of inertia and radius of gyration about the x-axis for the shaded area shown. Wherever possible, make expedient use of tabulated moments of inertia.





Solution. The composite area is composed of the positive area of the rectangle (1) and the negative areas of the quarter circle (2) and triangle (3). For the rectangle the moment of inertia about the x-axis, from Sample Problem A/1 (or Table D/3), is

$$I_x = \frac{1}{3}Ah^2 = \frac{1}{3}(80)(60)(60)^2 = 5.76(10^6)\ \text{mm}^4$$

From Sample Problem A/3 (or Table D/3), the moment of inertia of the negative quarter-circular area about its base axis x' is

$$I_{x'} = -\frac{1}{4}\left(\frac{\pi r^4}{4}\right) = -\frac{\pi}{16}(30)^4 = -0.1590(10^6)\ \text{mm}^4$$

We now transfer this result through the distance $\bar{r} = 4r/(3\pi) = 4(30)/(3\pi) = 12.73$ mm by the transfer-of-axis theorem to get the centroidal moment of inertia of part (2) (or use Table D/3 directly).

$$[\bar{I} = I - Ad^2] \qquad \bar{I}_x = -0.1590(10^6) - \left[-\frac{\pi(30)^2}{4}(12.73)^2\right]$$

$$= -0.0445(10^6)\ \text{mm}^4$$

The moment of inertia of the quarter-circular part about the x-axis is now

$$[I = \bar{I} + Ad^2] \qquad I_x = -0.0445(10^6) + \left[-\frac{\pi(30)^2}{4}\right](60 - 12.73)^2$$

$$= -1.624(10^6)\ \text{mm}^4$$

Finally, the moment of inertia of the negative triangular area (3) about its base, from Sample Problem A/2 (or Table D/3), is

$$I_x = -\frac{1}{12}bh^3 = -\frac{1}{12}(40)(30)^3 = -0.90(10^6)\ \text{mm}^4$$

The total moment of inertia about the x-axis of the composite area is, consequently,

$$I_x = 5.76(10^6) - 1.624(10^6) - 0.09(10^6) = 4.05(10^6)\ \text{mm}^4 \qquad \textit{Ans.}$$

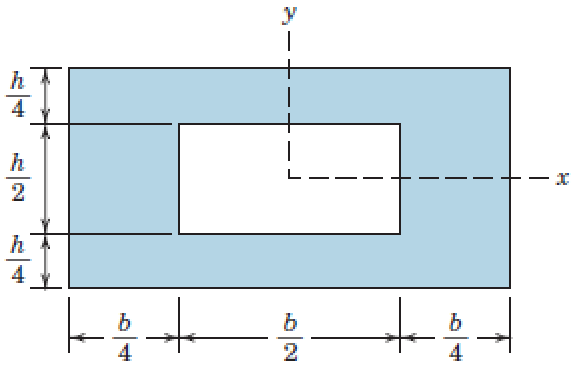
This result agrees with that of Sample Problem A/7.

The net area of the figure is $A = 60(80) - \frac{1}{4}\pi(30)^2 - \frac{1}{2}(40)(30) = 3490\ \text{mm}^2$ so that the radius of gyration about the x-axis is

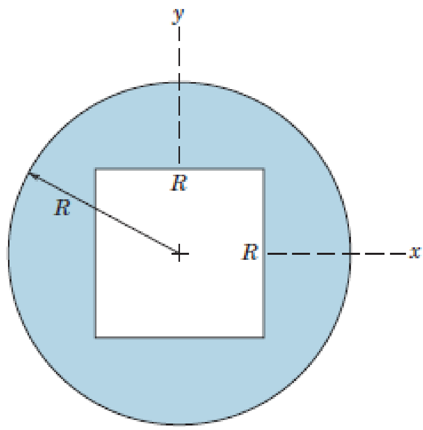
$$k_x = \sqrt{I_x/A} = \sqrt{4.05(10^6)/3490} = 34.0\ \text{mm} \qquad \textit{Ans.}$$

PROBLEMS

Q1/ Determine the moment of inertia about the x-axis of the rectangular area without and with the central rectangular hole



Q2/ Determine the moment of inertia about the y-axis of the circular area without and with the central square hole



Q3/Determine the percent reduction n in the polar moment of inertia of the square plate due to the introduction of the circular hole